Probabilistic Aspects of Default Risk Modeling

Tomasz Bielecki
Department of Mathematics, The Northeastern Illinois University, Chicago, USA
e-mail: T-Bielecki@neiu.edu

Marek Rutkowski
Institute of Mathematics, Technical University of Warsaw, Warszawa, Poland
e-mail: markrut@alpha.im.pw.edu.pl

Abstract

Various probabilistic techniques, which are used in the modeling of derivative securities (in particular, zero-coupon bonds) that are subject to default risk are presented in a systematic way. A large class of existing models of the defaultable term structure is covered by our analysis, in addition, some new ideas are presented.

1 Introduction

Let $B(t,T)$ and $D(t,T)$ denote time t prices of *default-free* and *default-risky* (or *defaultable*) zero-coupon bonds maturing at time T, respectively. The default-free bond pays \$1 at time T. The time T recovery value for the default-risky bond needs to be modeled. We are concerned with modeling the dynamics for the price process $D(t,T)$, as well as with relating $B(t,T)$ and $D(t,T)$. Our goal is to derive a model for default-free and defaultable term structures that can be statistically tested and that takes into account available data regarding: (i) migration probabilities between different credit rating classes (in particular probabilities of default), (ii) recovery rates for various credit rating classes, and (iii) credit spreads.

There are several existing models for the defaultable term structure, in which credit rating classes appear in an explicit way (see, for instance, Arvanitis et al. [2], Duffie and Singleton [24], Huge and Lando [30], Jarrow et al. [32], Lando [41], or Thomas et al. [60]). Apparently, all the above references seem not to account for at least one of the above itemized characteristics of risky debt. Additionally, the simplifying assumptions made, for instance, in Jarrow et al. [32] seem to be very restrictive (and probably not verifiable).

For a fixed horizon date $T^* > 0$, let $(\Omega, (\mathcal{F}_t)_{t\in[0,T^*]}, \mathbf{P})$ and $(\Omega, (\mathcal{F}_t)_{t\in[0,T^*]}, \mathbf{P}^*)$ denote the real-world and martingale probability spaces, respectively. The filtration $(\mathcal{F}_t)_{t\in[0,T^*]}$ is an enlargement of Wiener filtration, and is also accounting for random shocks leading to credit migration (credit quality jumps occuring typically at times when credit rating agencies produce their reports, or at random time of bankruptcy). The process r_t will denote the short-term interest rate, as usually. Typically, it is assumed that $\mathbf{P}^*$ is unique (or at least that a particular spot martingale measure $\mathbf{P}^*$ has already been chosen, if there are many spot martingale measures), so that defaultable claims can be priced through the standard risk-neutral valuation formula.

For instance, in Duffie and Singleton [22], the case of 'fractional recovery of market value' is examined. All relevant quantities are given in terms of an underlying Markov process Y, which represent 'state variables.' They derive the formula

$$D(t,T) = \mathbf{E}_{\mathbf{P}^*}\Big\{ \exp\Big(- \int_t^T R_u \, du \Big) \,\Big|\, \mathcal{G}_t \Big\}, \tag{1}$$

with $R_t = \rho(Y_t)$ is the 'default-adjusted' short-term interest rate. Conditioning is with respect to the filtration is generated by the state-variables process Y; that is, $\mathcal{G}_t = \sigma(Y_u \,:\, u \le t)$.

On the other hand, Jarrow and Turnbull [34] and Jarrow et al. [32] focus on the 'fractional recovery of par,' so that

$$D(t,T) = \mathbf{E}_{\mathbf{P}^*}\Big\{ \exp\Big(- \int_t^T r_u \, du \Big) X_T \,\Big|\, \mathcal{F}_t \Big\}, \tag{2}$$

where $X_T = \delta \, \mathrm{I}_{\{T \ge \tau\}} + \mathrm{I}_{\{T < \tau\}}$. Here τ is the time at which bankruptcy occurs, and δ is the (constant) recovery rate. A similar approach is taken in Madan and Unal [46].

Let's look at things from the HJM perspective. Towards this end, suppose that there are K credit classes or states, the K^{th} state denoting the state of default. Thus, the risky bond can be in any of the states $i \in \mathcal{K} = \{1, \dots, K\}$. Let C denote the Markov process taking values in $\mathcal{K} \times \mathcal{K}$. It is convenient to assume that the state K is absorbing. The process will model migration between credit classes. So, for example,

$$\mathbf{P}\big\{ C_{t+s} = (i,j) \,|\, \sigma(C_u \,:\, u \le t) \big\} = \mathbf{P}\big\{ C_{t+s} = j \,|\, C_t = (C_t^1, C_t^2) \big\}$$

provides the probability that the bond is in the credit class j at time $t+s$, and the immediately preceding bond's class was i given the bond was in the credit class C_t^2 at time t which was immediately preceded by class C_t^1. Such probabilities can be estimated from available data (CreditMetrics does not do this directly though). This idea of 'twin state' Markov chain is similar to what is suggested in Arvantis et al. [2]. Note that the jump times for the Markov chain C_t can be random, as well as deterministic (if in fact changes in credit ratings occur only at selected deterministic times during the lifetime of the bond: the times at which rating agencies present their reports).

We wish to model the price process of a defaultable bond, for a given initial condition C_0 at time 0 (we assume that $C_0^2 \ne K$). We shall focus on the case of the fractional recovery of par. Let $\delta_i \in [0,1)$ be the recovery rate for a bond which belonged to class i just before default occurred. To be more specific, if T-maturity unit bond defaults before or at time T, its owner is entitled to the payoff δ_i at time T, provided that the bond belonged to class i just before default occurred (the estimates of δ_i's are regularly done by CreditMetrics).

Let the instantaneous forward rates be $f(t,T)$ (for the default-free bond), so that the price $B(t,T)$ of a default-free zero-coupon bond equals

$$B(t,T) = \exp\Big(- \int_t^T f(t,u) \, du \Big). \tag{3}$$

Similarly, for any $i < K$, we write $g_i(t,T)$ the 'conditional' instantaneous forward rates for the risky bond that is in state i at time t. It should be stressed the process

$$D_i(t,T) := \exp\Big(- \int_t^T g_i(t,u) \, du \Big) \tag{4}$$

does not represent the price process of tradable security. In other words, 'the risky bond that is in state i at time t' is not a tradable asset. In the present framework, a particular defaultable bond is formally defined by its principal value (assumed to be 1), maturity T, and the initial condition C_0.

We postulate that the price process of a defaultable bond, for a given initial condition C_0, is given by the following expression

$$D_{C_t}(t,T) := \mathrm{I}_{\{C_t^2\neq K\}} \exp\Big(-\int_t^T g_{C_t^2}(t,u)\,du\Big) + \delta_{C_t^1}\,\mathrm{I}_{\{C_t^2=K\}} \exp\Big(-\int_t^T f(t,u)\,du\Big) \tag{5}$$

for every $t\in[0,T]$. Put another way,

$$D_{C_t}(t,T) = \mathrm{I}_{\{C_t^2\neq K\}} D_{C_t^2}(t,T) + \delta_{C_t^1}\,\mathrm{I}_{\{C_t^2=K\}} B(t,T). \tag{6}$$

Therefore, for any initial condition C_0, at any time t we have

$$D_{C_t}(t,T) = D_i(t,T) \quad \text{on the set} \quad \{C_t^2 = i\}$$

for every $i<K$. Furthermore,

$$D_{C_t}(t,T) = \delta_i B(t,T) \quad \text{on the set} \quad \{(C_t^1,C_t^2)=(i,K)\}.$$

We thus see that $D_i(t,T)$ does indeed represent the price at time t of a default-risky bond, provided that it is currently in the i^{th} credit class. Put another way, $D_i(t,T)$ is the price of a defaultable bond, conditional on the event 'the bond is currently in the credit class $i<K$.' Due to the Markovian structure of the model, the value $D_i(t,T)$ does not depend on the history of a particular defaultable bond, so that we have a unique price for all defaultable bonds which are currently in a given credit class.

For any i, we define the i^{th} *credit spread process* $\gamma_i(t,u)$ by setting

$$\gamma_i(t,u) = g_i(t,u) - f(t,u).$$

Combining (3) with (4), we get

$$D_i(t,T) = B(t,T)\exp\Big(-\int_t^T \gamma_i(t,u)\,du\Big).$$

Also

$$D_{C_t}(t,T) = B(t,T)\Big\{\mathrm{I}_{\{C_t^2\neq K\}}\exp\Big(-\int_t^T \gamma_{C_t^2}(t,u)\,du\Big) + \delta_{C_t^1}\,\mathrm{I}_{\{C_t^2=K\}}\Big\}. \tag{7}$$

To simplify formulae (5) and (7), it is convenient to denote $f(t,T)=g_K(t,T)$, so that $\gamma_K(t,u)=0$. Then (5) and (7) become

$$D_{C_t}(t,T) = X_t\exp\Big(-\int_t^T g_{C_t^2}(t,u)\,du\Big),$$

and

$$D_{C_t}(t,T) = B(t,T)X_t\exp\Big(-\int_t^T \gamma_{C_t^2}(t,u)\,du\Big),$$

respectively, where X_t is the promised payoff, as at time t,

$$X_t = \mathrm{I}_{\{C_t^2\neq K\}} + \delta_{C_t^1}\,\mathrm{I}_{\{C_t^2=K\}}.$$

Finally let us introduce the *default time* by setting (as usual, given the initial condition C_0)

$$\tau = \inf\{\, t\in\mathbf{R}_+ \,:\, C_t^2 = K\,\}. \tag{8}$$

Then X_t satisfies

$$X_t = \mathrm{I}_{\{t<\tau\}} + \delta_{C_t^1}\,\mathrm{I}_{\{t\geq\tau\}}.$$

In the case of zero recovery, i.e., when $\delta_i=0$ for every $i<K$, we obtain (cf. (6))

$$D_{C_t}(t,T) = \mathrm{I}_{\{t<\tau\}}\exp\Big(-\int_t^T g_{C_t^2}(t,u)\,du\Big) = \mathrm{I}_{\{t<\tau\}}B(t,T)\exp\Big(-\int_t^T \gamma_{C_t^2}(t,u)\,du\Big).$$

Suppose, on the contrary, that $\delta_1=1$ for every $i<K$. We then expect to have $D_{X_t}(t,T)=B(t,T)$ for every t. For this to hold, it is enough to assume that for any i the 'conditional' forward rate $g_i(t,T)$ coincides with $f(t,T)$.

2 Default Time

The exposition in this section is mainly based on Duffie et al. [21]. In this section, our goal is to present the most fundamental results which can be obtained using the intensity-based approach. In Section 3, special attention will be paid to the various kinds of recovery rates, such as, for instance, zero recovery, fractional recovery of par, and fractional recovery of market value. On the other hand, in order to obtain as explicit valuation formulae as possible, we shall still assume that only two states are possible, namely, non-default and default. An analysis of the case of several credit rating classes is postponed to Sections 4–5. Unless stated otherwise, we make the following standing assumptions.

(A.1) For a finite horizon date T^*, we are given a filtered probability space $(\Omega, (\mathcal{F}_t)_{t \in [0,T^*]}, \mathbf{P}^*)$, with the probability measure $\mathbf{P}^*$ interpreted as a martingale measure for our underlying market model (that is, a complete or incomplete securities market model, which we do not need to specify in more details at this stage).

(A.2) For a given default-risky security, its *default process* is modeled through a count process N with strictly positive intensity process λ under $\mathbf{P}^*$. More specifically, the *default time* τ is a stopping time, defined as the first jump time of the default process N. The following property of τ is crucial: if we introduce the first jump process $H_t = \mathrm{I}_{\{t \geq \tau\}}$, then the compensated process

$$M_t = H_t - \int_0^t \lambda_u \, \mathrm{I}_{\{u \leq \tau\}} \, du = H_t - \int_0^t h_u \, du, \quad \forall \, t \in [0, T^*], \tag{9}$$

follows a martingale under $\mathbf{P}^*$.

(A.3) Given a maturity date $T > 0$, a $\mathcal{F}_T$-measurable random variable X represents the *promised claim,* that is, the amount of cash which the owner of a defaultable claim is entitled to receive at time T, provided that the default has not occured before the maturity date T.

(A.4) A predictable process Z models the payoff which is actually received by the owner of a defaultable claim, if default occurs before maturity T. At this stage, no specific relationships between Z and X are assumed, however, we shall refer to Z as the *recovery process* of X.

(A.5) An adapted process r stands for the short-term interest rate, and B_t, $t \in [0, T^*]$, is the associated savings account process.

In view of recent developments in term structure modeling, the last assumption may seem to be slightly 'old-fashioned.' It should be stressed that Assumption (A.5) is not essential (it would be perhaps instructive to replace B_t by $B(t,T)$ and to interpret $\mathbf{P}^*$ as the forward measure). The main result in the intensity-based approach states that a defaultable security can be priced as if it were a default-risk free security, provided that the credit spread is already incorporated in the risk premium. In other words, the risk premium process of a defaultable security differs from that associated with a risk-free bond, both in the real-world and in the risk-neutral world. In particular, in a risk-neutral world the risk premium associated with a risk-free bond vanishes, and the risk premium associated with a defaultable security is still present. Such a statement is equally meaningful in any arbitrage-free framework, so that Assumption (A.5) was introduced for the sake of convenience only.

Example 2.1 If the intensity process $\lambda_t = \lambda > 0$ is constant, the process H can be seen as a continuous-time Markov chain with the state space $\{0, 1\}$, and with constant intensity matrix $\Lambda = [\lambda_{ij}]_{0 \leq i,j \leq 1}$, where $\lambda_{00} = -\lambda$, $\lambda_{01} = \lambda$, and $\lambda_{1i} = 0$ for $i = 0, 1$ (so that the state 1 is absorbing). In this case, N follows a standard Poisson process with constant intensity λ. This will be later generalized in two directions. First, in some circumstances it might be natural to assume that $\lambda_t = \lambda(Y_t)$, where Y is a given k-dimensional adapted stochastic process, and $\lambda : \mathbf{R}^k \to \mathbf{R}_+$ is a strictly positive deterministic function. Second, the basic model can be extended to accomodate for different credit rating classes, $\Lambda_t = [\lambda_{ij}(Y_t)]_{0 \leq i,j \leq K}$, with K being an absorbing state (see, for instance, Jarrow et al. [32]) or Section 4 below).

We need first to formally define the value process S of a *defaultable claim,* represented by a triplet (X, Z, τ). Since we assume throughout that $\mathbf{P}^*$ is a spot martingale measure, it is natural to postulate that the value S_0 at time 0 of a defaultable claim (X, Z, τ) equals (we may and do assume that $\mathbf{P}^*\{\tau = 0\} = 0$)

$$S_0 := B_0 \, \mathbf{E}_{\mathbf{P}^*}\Big(\int_{]0,T]} B_u^{-1} \, dD_u \Big), \tag{10}$$

where B stands for the risk-free savings account, and D is the 'dividend process' (cf. (A.3)–(A.4))

$$D_t = \int_{]0,t]} Z_u \, dH_u + X(1 - H_T) \, \mathrm{I}_{\{t=T\}}. \tag{11}$$

More generally, under Assumption (A.2) we have $\mathbf{P}^*\{\tau = t\} = 0$ for any t, so that (10) can be easily generalized to give

$$S_t := B_t \, \mathbf{E}_{\mathbf{P}^*}\Big(\int_{]t,T]} B_u^{-1} \, dD_u \,\Big|\, \mathcal{F}_t \Big), \tag{12}$$

or equivalently,

$$S_t := B_t \, \mathbf{E}_{\mathbf{P}^*}\Big(\int_{]t,T]} B_u^{-1} Z_u \, dH_u + B_T^{-1} X \, \mathrm{I}_{\{T<\tau\}} \,\Big|\, \mathcal{F}_t \Big). \tag{13}$$

In particular, at maturity of the contract we have $S_T = X \, \mathrm{I}_{\{T<\tau\}}$, as expected. Notice that (13) can be also rewritten as follows

$$S_t = B_t \, \mathbf{E}_{\mathbf{P}^*}\Big(B_\tau^{-1} Z_\tau \, \mathrm{I}_{\{t<\tau\le T\}} + B_T^{-1} X \, \mathrm{I}_{\{T<\tau\}} \,\Big|\, \mathcal{F}_t \Big), \tag{14}$$

or more explicitly,

$$S_t = \mathbf{E}_{\mathbf{P}^*}\Big\{ \exp\Big(- \int_t^{\tau \wedge T} r_u \, du \Big) \big(Z_\tau \, \mathrm{I}_{\{t<\tau\le T\}} + X \, \mathrm{I}_{\{T<\tau\}} \big) \,\Big|\, \mathcal{F}_t \Big\}. \tag{15}$$

Definition 2.1 By a *defaultable claim* we mean a triplet (X, Z, τ), where X is the *promised payoff,* Z represents the *recovery process* of X, and τ is the *default time.* The *price* (or *value*) *process* S of a defaultable claim (X, Z, τ) is given by either of the formulae (12)–(15).

Remark 2.1 Notice that Definition 2.1 specifies the price of a defaultable security on the ex-dividend basis. In particular, for any t we have $S_t = 0$ on the event $\{\tau < t\}$. Intuitively, this means that the payoff at the event of default is received in cash (and invested, e.g., in the risk-free savings account), and the defaultable security becomes worthless forever. This convention agrees, of course, with our current set of assumptions (A.1)–(A.5), but does not necessarily reflect the actual bankruptcy procedures. Once again, it should be generalized to fit more adequately the real-world behaviour of defaultable securities.

The following lemma provides still another representation for the price process S of a defaultable claim. It appears that, due to Assumption (A.2), the integration with respect to the process H_t can be replaced with the integration with respect to its *intensity measure* $h_t \, dt$. The presence in (16)–(17) of the complement of the default event $\{\tau \le T\}$ shows, however, that the default time τ was not yet completely eliminated from the valuation formula.

Lemma 2.1 *The price process S admits the following representations*

$$S_t = B_t \, \mathbf{E}_{\mathbf{P}^*}\Big(\int_t^T B_u^{-1} Z_u h_u \, du + B_T^{-1} X \, \mathrm{I}_{\{T<\tau\}} \,\Big|\, \mathcal{F}_t \Big) \tag{16}$$

and

$$S_t = \mathbf{E}_{\mathbf{P}^*}\Big(\int_t^T (Z_u h_u - r_u S_u) \, du + X \, \mathrm{I}_{\{T<\tau\}} \,\Big|\, \mathcal{F}_t \Big). \tag{17}$$

Proof. The first formula follows from (13), combined with the equality

$$\mathbf{E}_{\mathbf{P}^*}\Big(\int_{]t,T]} B_u^{-1} Z_u \, dH_u \,\Big|\, \mathcal{F}_t\Big) = \mathbf{E}_{\mathbf{P}^*}\Big(\int_{]t,T]} B_u^{-1} Z_u \left(dM_u + h_u \, du\right)\Big|\, \mathcal{F}_t\Big)$$

which in turn is an immediate consequence of (9). For the second, it is enough to rewrite (16) as follows

$$S_t = B_t\Big(\tilde{M}_t - \int_0^t B_u^{-1} Z_u h_u \, du\Big), \tag{18}$$

where we have put

$$\tilde{M}_t = \mathbf{E}_{\mathbf{P}^*}\Big(\int_0^T B_u^{-1} Z_u h_u \, du + B_T^{-1} X \, \mathrm{I}_{\{T<\tau\}} \,\Big|\, \mathcal{F}_t\Big).$$

Applying Itô's formula to (18), we obtain

$$dS_t = (r_t S_t - Z_t h_t)\, dt + B_t \, d\tilde{M}_t,$$

and thus

$$\mathbf{E}_{\mathbf{P}^*}(S_T \,|\, \mathcal{F}_t) = S_t + \mathbf{E}_{\mathbf{P}^*}\Big(\int_t^T \left(r_u S_u - Z_u h_u\right) du \Big|\, \mathcal{F}_t\Big).$$

Since $S_T = X \, \mathrm{I}_{\{T<\tau\}}$, the last equality yields (17). □

The following result – due to Duffie et al. [21] – plays a crucial role in what follows.

Proposition 2.1 *For a given process Z, we define the process V by setting*

$$V_t = \tilde{B}_t \, \mathbf{E}_{\mathbf{P}^*}\Big(\int_t^T \tilde{B}_u^{-1} Z_u \lambda_u \, du + \tilde{B}_T^{-1} X \,\Big|\, \mathcal{F}_t\Big), \tag{19}$$

where $\tilde{B}$ is the 'savings account' corresponding to the default-adjusted short-term rate $R_t = r_t + \lambda_t$, that is,

$$\tilde{B}_t = \exp\Big(\int_0^t (r_u + \lambda_u)\, du\Big). \tag{20}$$

Then

$$\mathrm{I}_{\{t<\tau\}} V_t = B_t \, \mathbf{E}_{\mathbf{P}^*}\Big(B_\tau^{-1}(Z_\tau + \Delta V_\tau)\, \mathrm{I}_{\{t<\tau\le T\}} + B_T^{-1} X \, \mathrm{I}_{\{T<\tau\}} \,\Big|\, \mathcal{F}_t\Big). \tag{21}$$

Proof. In view of (19) we have

$$V_t = \tilde{B}_t\Big(N_t - \int_0^t \tilde{B}_u^{-1} Z_u \lambda_u \, du\Big), \tag{22}$$

where N is a martingale given by the formula

$$N_t = \mathbf{E}_{\mathbf{P}^*}\Big(\int_0^T \tilde{B}_u^{-1} Z_u \lambda_u \, du + \tilde{B}_T^{-1} X \,\Big|\, \mathcal{F}_t\Big). \tag{23}$$

Using Itô's product rule, we obtain

$$dV_t = r_t V_t \, dt - (Z_t - V_{t-})\lambda_t \, dt + \tilde{B}_t \, dN_t. \tag{24}$$

Define $U_t = \tilde{H}_t V_t$, where $\tilde{H}_t = 1 - H_t = \mathrm{I}_{\{t<\tau\}}$, so that $U_t = \mathrm{I}_{\{t<\tau\}} V_t$. On the one hand, it is useful to observe that (21) may be rewritten as follows

$$U_t = B_t \, \mathbf{E}_{\mathbf{P}^*}\Big(\int_{]t,T]} B_u^{-1}(Z_u + \Delta V_u)\, dH_u + B_T^{-1} X \, \mathrm{I}_{\{T<\tau\}} \,\Big|\, \mathcal{F}_t\Big). \tag{25}$$

On the other hand, an application of Itô's product rule yields (the process $\tilde{H}$ is of finite variation, and $\tilde{H}_{t-} = \tilde{H}_t$)

$$dU_t = d(V_t\tilde{H}_t) = \tilde{H}_t\,dV_t + V_{t-}\,d\tilde{H}_t + \Delta V_t \Delta \tilde{H}_t.$$

In view of (24) and equality $\lambda_t \,\mathrm{I}_{\{t\le\tau\}} = h_t$, this yields

$$dU_t = d(V_t\tilde{H}_t) = \tilde{H}_t\big(r_tV_t\,dt - (Z_t - V_{t-})h_t\,dt + \tilde{B}_t\,dN_t\big) + V_{t-}\,d\tilde{H}_t + \Delta V_t \Delta \tilde{H}_t.$$

After rearranging and noting that $\Delta\tilde{H}_t = -\Delta H_t$, we obtain

$$dU_t = r_tU_t\,dt - (Z_t + \Delta V_t)\,dH_t + \tilde{N}_t, \tag{26}$$

where $\tilde{N}_t$ stands for the local martingale, more precisely,

$$d\tilde{N}_t = \tilde{H}_t\tilde{B}_t\,dN_t + (V_{t-} - Z_t)\,dM_t.$$

Since $U_T = X\,\mathrm{I}_{\{T<\tau\}}$, (26) gives (25) (if the local martingale $\tilde{N}$ is in fact a 'true' martingale). □

Corollary 2.1 *Let the processes S and V be defined by* (13) *and* (19), *respectively. Then* (i)

$$S_t = \mathrm{I}_{\{t<\tau\}}\Big\{V_t - B_t\,\mathbf{E}_{\mathbf{P}^*}\big(B_\tau^{-1}\,\mathrm{I}_{\{\tau\le T\}}\Delta V_\tau \,\big|\, \mathcal{F}_t\big)\Big\}, \tag{27}$$

(ii) *if $\Delta V_\tau = 0$, then $S_t = \mathrm{I}_{\{t<\tau\}}V_t$ for every $t\in[0,T]$.*

Proof. A comparison of (14) and (21) yields

$$S_t = U_t - B_t\,\mathbf{E}_{\mathbf{P}^*}\big(B_\tau^{-1}\,\mathrm{I}_{\{t<\tau\le T\}}\Delta V_\tau \,\big|\, \mathcal{F}_t\big).$$

Formula (27) now easily follows. □

For easy further reference, we shall write down the particular case of (27) when $\Delta V_\tau = 0$. In this case, we have simply $S_t = U_t$, that is,

$$S_t = \mathrm{I}_{\{t<\tau\}}\tilde{B}_t\,\mathbf{E}_{\mathbf{P}^*}\Big(\int_t^T \tilde{B}_u^{-1}Z_u\lambda_u\,du + \tilde{B}_T^{-1}X \,\Big|\, \mathcal{F}_t\Big). \tag{28}$$

In view of the relationship established in part (ii) of Corollary 2.1, the process V given by formula (19) is commonly referred to as the *pre-default value* of a defaultable claim X.

3 Recovery Rates

In this section, we shall further specify the model presented in the previous section, by introducing various kinds of recovery processes.

3.1 Exogenous Recovery Rates

We assume that Z is exogenously given predictable process. In this case, the price process S is uniquely defined through (13) so that no ambiguity may arise. Notice however, that in formula (14) only the values of the process Z at default time τ are essential. This suggests that instead of specifying the process Z, we could instead predetermine only the $\mathcal{F}_\tau$-measurable random variable Z_τ. Let us then postulate that we are given a bounded $\mathcal{F}_\tau$-measurable random variable W, which models the recovery rate at default time. The price process of a defaultable claim X now equals (as before, we assume that a defaultable claim settles at time T)

$$S_t := B_t\,\mathbf{E}_{\mathbf{P}^*}\big(B_\tau^{-1}W\,\mathrm{I}_{\{t<\tau\le T\}} + B_T^{-1}X\,\mathrm{I}_{\{T<\tau\}} \,\big|\, \mathcal{F}_t\big). \tag{29}$$

Following Duffie [18], we shall now consider both the case of discrete-time and continual recovery of a defaultable claim with recovery rate W. As expected, the case of *continual recovery* corresponds to expression (29). It appears (see Duffie [17] in this regard) that the results of Section 2 remain valid in the case of continual recovery with recovery rate W, provided that we replace the recovery process Z with the compensator of W (under $\mathbf{P}^*$). By definition, a *compensator* of W is a predictable process, $\widetilde{W}$ say, which satisfies $\widetilde{W}_{\tau-} = \mathbf{E}_{\mathbf{P}^*}(W \,|\, \mathcal{F}_{\tau-})$.

A *discrete-time recovery* assumes that the payoff at the event of default is received by the owner of a claim on the first date after default among a predetermined set of *admissible* dates $0 = T_0 < T_1 < \ldots < T_n = T$. Under this convention, the valuation formula reads

$$\tilde{S}_t := \sum_{T_i \geq t} B_t \, \mathbf{E}_{\mathbf{P}^*}\big(B_{T_i}^{-1} W \, \mathrm{I}_{\{T_{i-1} < \tau \leq T_i\}} \,\big|\, \mathcal{F}_t\big) + B_t \, \mathbf{E}_{\mathbf{P}^*}\big(B_T^{-1} X \, \mathrm{I}_{\{T < \tau\}} \,\big|\, \mathcal{F}_t\big). \tag{30}$$

In practical terms, when default occurs, the associated payoff (if any) is postponed to the nearest date T_i after default. It should be stressed that it is now enough to assume that a random variable W is such that for every $i = 1, \ldots, n$, the random variable $W_i = W \, \mathrm{I}_{\{T_{i-1} < \tau \leq T_i\}}$ is $\mathcal{F}_{T_i}$-measurable. In other words, the amount paid off at the date T_i is based on the total information which is available at this time, including the default event $\{T_{i-1} < \tau \leq T_i\}$.

It is interesting to notice that the valuation formula (30) has slightly different features than our basic valuation formula (13). Indeed, formula (13) implicitly assumes that a defaultable claim becomes worthless as soon as a default occurs. On the other hand, when formula (30) is used to value a defaultable claim, a claim becomes worthless not at the time of default, but after the nearest date from the set of admissible dates.

Our next goal is to give formula (30) a more explicit representation. For a fixed $t \leq T$, we shall write $i_0 = i_0(t) = \inf\{\, i \,:\, T_i \geq t \,\}$. It is thus clear that

$$\tilde{S}_t = \sum_{i=i_0}^{n} (\hat{U}_t^i - \tilde{U}_t^i) + U_t^n,$$

where

$$\hat{U}_t^i = B_t \, \mathbf{E}_{\mathbf{P}^*}\big(B_{T_i}^{-1} W_i \, \mathrm{I}_{\{T_{i-1} < \tau\}} \,\big|\, \mathcal{F}_t\big), \quad \tilde{U}_t^i = B_t \, \mathbf{E}_{\mathbf{P}^*}\big(B_{T_i}^{-1} W_i \, \mathrm{I}_{\{T_i < \tau\}} \,\big|\, \mathcal{F}_t\big),$$

and

$$U_t^n = B_t \, \mathbf{E}_{\mathbf{P}^*}\big(B_{T_n}^{-1} X \, \mathrm{I}_{\{T_n < \tau\}} \,\big|\, \mathcal{F}_t\big).$$

Since for every $i = i_0, \ldots, n$ we have: (a) $\mathcal{F}_t \subset \mathcal{F}_{T_i}$, and (b) the random variable W_i is $\mathcal{F}_{T_i}$-measurable, the evaluation of $\tilde{U}_t^i$, $i = 1, \ldots, n$ and U_t^n is standard. Indeed, we may apply previously established results, with $Z = 0$ and $T = T_i$. To get a more transparent expression for the valuation formula, we shall assume that $\Delta V_\tau = 0$, where V stands for the pre-default value process introduced in Proposition 2.1 (of course, in the present context V depends on i, so that the assumption that V doesn't jump at default time is made for every i). In view of Corollary 2.1, we have

$$\tilde{U}_t^i = \mathrm{I}_{\{t < \tau\}} \, \tilde{B}_t \, \mathbf{E}_{\mathbf{P}^*}\big(\tilde{B}_{T_i}^{-1} W_i \,\big|\, \mathcal{F}_t\big), \quad \forall\, i = 1, \ldots, n, \quad U_t^n = \mathrm{I}_{\{t < \tau\}} \, \tilde{B}_t \, \mathbf{E}_{\mathbf{P}^*}\big(\tilde{B}_{T_i}^{-1} X \,\big|\, \mathcal{F}_t\big).$$

We may proceed in a similar way when dealing with $\hat{U}_t^i$, provided that $i \geq i_0 + 1$ (this ensures that $\mathcal{F}_t \subset \mathcal{F}_{T_{i-1}}$). To this end, we find it convenient to represent $\hat{U}_t^i$ as follows

$$\hat{U}_t^i = B_t \, \mathbf{E}_{\mathbf{P}^*}\Big(B_{T_{i-1}}^{-1} \mathbf{E}_{\mathbf{P}^*}\big(B_{T_{i-1}} B_{T_i}^{-1} W_i \,\big|\, \mathcal{F}_{T_{i-1}}\big) \, \mathrm{I}_{\{T_{i-1} < \tau\}} \,\Big|\, \mathcal{F}_t\Big).$$

This means that

$$\hat{U}_t^i = B_t \, \mathbf{E}_{\mathbf{P}^*}(B_{T_{i-1}}^{-1} Y_i \, \mathrm{I}_{\{T_{i-1} < \tau\}} \,|\, \mathcal{F}_t),$$

where Y_i is a $\mathcal{F}_{T_{i-1}}$-measurable random variable

$$Y_i = B_{T_{i-1}} \mathbf{E}_{\mathbf{P}^*}(B_{T_i}^{-1} W_i \,|\, \mathcal{F}_{T_{i-1}}). \tag{31}$$

Notice that Y_i represents the price at time T_{i-1} of a non-defaultable claim that pays W_i at time T_i. Arguing along the same lines as before, we get

$$\hat{U}^i_t = \mathrm{I}_{\{t<\tau\}}\, \tilde{B}_t\, \mathbf{E}_{\mathbf{P}^*}\big(\tilde{B}^{-1}_{T_{i-1}} Y_i \,\big|\, \mathcal{F}_t\big).$$

It thus remains to analyze the following term

$$\hat{U}^{i_0}_t = B_t\, \mathbf{E}_{\mathbf{P}^*}\Big(\mathbf{E}_{\mathbf{P}^*}\big(B^{-1}_{T_{i_0}} W_{i_0} \,\big|\, \mathcal{F}_{T_{i_0-1}}\big)\, \mathrm{I}_{\{T_{i_0-1}<\tau\}} \,\Big|\, \mathcal{F}_t\Big).$$

Since $\mathcal{F}_{T_{i_0}} \subset \mathcal{F}_t$ and the event $\{T_{i_0-1} < \tau\}$ belongs to $\mathcal{F}_{T_{i_0}}$, we obtain

$$\hat{U}^{i_0}_t = \mathrm{I}_{\{T_{i_0-1}<\tau\}} B_t\, \mathbf{E}_{\mathbf{P}^*}\big(B^{-1}_{T_{i_0}} W_{i_0} \,\big|\, \mathcal{F}_t\big) = \mathrm{I}_{\{T_{i_0-1}<\tau\}} Y_{i_0},$$

where Y_{i_0} represents the price at time t of a non-defaultable claim that pays W_{i_0} at time T_{i_0}. We are in a position to state the following result. Let us stress that we assume that part (ii) in Corollary 2.1 may be applied to each term $\hat{U}^i_t$ and $\tilde{U}^i_t$.

Proposition 3.1 *Let the price $\tilde{S}_t$ at time $t \le T$ of a defaultable claim W with discrete-time recovery be given by formula* (30). *Then*

$$\begin{aligned}
\tilde{S}_t \;&=\; \mathrm{I}_{\{T_{i_0-1}<\tau\}} B_t\, \mathbf{E}_{\mathbf{P}^*}\big(B^{-1}_{T_{i_0}} W_{i_0} \,\big|\, \mathcal{F}_t\big) + \mathrm{I}_{\{t<\tau\}} \sum_{i=i_0+1}^{n} \tilde{B}_t\, \mathbf{E}_{\mathbf{P}^*}\big(\tilde{B}^{-1}_{T_{i-1}} Y_i \,\big|\, \mathcal{F}_t\big) \\
&\quad - \mathrm{I}_{\{t<\tau\}} \sum_{i=i_0}^{n} \tilde{B}_t\, \mathbf{E}_{\mathbf{P}^*}\big(\tilde{B}^{-1}_{T_i} W_i \,\big|\, \mathcal{F}_t\big) + \mathrm{I}_{\{t<\tau\}}\, \tilde{B}_t\, \mathbf{E}_{\mathbf{P}^*}\big(\tilde{B}^{-1}_{T_n} X \,\big|\, \mathcal{F}_t\big),
\end{aligned}$$

where $i_0 = i_0(t) = \inf\{\, i \,:\, T_i > t\}$, $W_i = W\,\mathrm{I}_{\{T_{i-1}<\tau\le T_i\}}$, Y_i is given by (31)*, and $\tilde{B}$ by* (20).

We shall now focus on the case of a defaultable term structure, that is, we set $X = 1$. The most tractable cases are: (i) the case of zero recovery: $W = 0$, (ii) the case of fractional recovery of par: $W = \delta$ with $0 < \delta < 1$ (in principle, δ can be any real number). For any adapted process γ, we find it convenient to denote

$$B^{\gamma}(t,T) = \mathbf{E}_{\mathbf{P}^*}\Big\{ \exp\Big(-\int_t^T (r_u + \gamma_u)\, du\Big) \,\Big|\, \mathcal{F}_t \Big\}. \tag{32}$$

Notice that $B^0(t,T) = B(t,T)$, and $B^{\gamma}(t,T) < B(t,T)$ if γ is strictly positive.

3.1.1 Zero recovery. In the case of zero recovery, formulae (29) and (30) yield, as expected, the same result for the price process $D^0(t,T)$ of the T-maturity defaultable bond. Namely, we have

$$D^0(t,T) = B_t\, \mathbf{E}_{\mathbf{P}^*}(B_T^{-1}\, \mathrm{I}_{\{T<\tau\}} \,|\, \mathcal{F}_t). \tag{33}$$

Suppose that we are in a position to apply part (ii) in Corollary 2.1 (i.e. $\Delta V_{\tau} = 0$). Then

$$D^0(t,T) = \mathrm{I}_{\{t<\tau\}} \tilde{B}_t\, \mathbf{E}_{\mathbf{P}^*}(\tilde{B}_T^{-1} \,|\, \mathcal{F}_t) = \mathrm{I}_{\{t<\tau\}} B^{\lambda}(t,T).$$

This means that the price of a bond before default can be calculated in a 'standard' way, provided that the risk-free rate r is substituted with the default-adjusted rate $R = r + \lambda$. In particular, if λ is strictly positive then $D^0(t,T) < B(t,T)$ for $t < T$, and $D^0(T,T) \le B(T,T) = 1$.

3.1.2 Fractional recovery of par. In the case of a non-zero recovery coefficient δ, for the price $D^{\delta}(t,T)$ of a defaultable bond with continual recovery we get

$$D^{\delta}(t,T) = B_t\, \mathbf{E}_{\mathbf{P}^*}\big(\delta B_{\tau}^{-1}\, \mathrm{I}_{\{t<\tau\le T\}} + B_T^{-1}\, \mathrm{I}_{\{T<\tau\}} \big| \mathcal{F}_t\big) = \mathrm{I}_{\{t<\tau\}} \tilde{B}_t\, \mathbf{E}_{\mathbf{P}^*}\Big(\delta \int_t^T \tilde{B}_u^{-1} \lambda_u\, du + \tilde{B}_T^{-1} \Big| \mathcal{F}_t\Big),$$

where the second equality holds when $\Delta V_\tau = 0$. The price of a defaultable bond with discrete-time recovery, $\tilde{D}^\delta(t,T)$ say, equals (cf. (30))

$$\tilde{D}^\delta(t,T) = \sum_{T_i \geq t} B_t \, \mathbf{E}_{\mathbf{P}^*}\big(\delta B_{T_i}^{-1} \mathrm{I}_{\{T_{i-1} < \tau \leq T_i\}} \,\big|\, \mathcal{F}_t\big) + B_t \, \mathbf{E}_{\mathbf{P}^*}\big(B_T^{-1} \mathrm{I}_{\{T < \tau\}} \,\big|\, \mathcal{F}_t\big).$$

Let us analyze the latter case in more details. Suppose that $T_{i_0-1} \leq t < T_{i_0}$. First, we have

$$\tilde{D}^\delta(t,T) = \delta B_t \sum_{i=i_0}^{n} \Big(\mathbf{E}_{\mathbf{P}^*}\big(B_{T_i}^{-1} \mathrm{I}_{\{T_{i-1} < \tau\}} \,\big|\, \mathcal{F}_t\big) - \mathbf{E}_{\mathbf{P}^*}\big(B_{T_i}^{-1} \mathrm{I}_{\{T_i < \tau\}} \,\big|\, \mathcal{F}_t\big)\Big) + B_t \, \mathbf{E}_{\mathbf{P}^*}\big(B_{T_n}^{-1} \mathrm{I}_{\{T_n < \tau\}} \,\big|\, \mathcal{F}_t\big),$$

or in an abbreviated form

$$\tilde{D}^\delta(t,T) = \sum_{i=i_0}^{n} \delta \, \hat{U}(t,T_i) - \sum_{i=i_0}^{n} \delta \, \tilde{U}(t,T_i) + U(t,T_n). \tag{34}$$

Since $T_{i_0-1} \leq t$ and thus $\mathcal{F}_{T_{i_0-1}} \subset \mathcal{F}_t$, it is clear that

$$\hat{U}(t,T_{i_0}) = B_t \, \mathbf{E}_{\mathbf{P}^*}\big(B_{T_{i_0}}^{-1} \mathrm{I}_{\{T_{i_0-1} < \tau\}} \,\big|\, \mathcal{F}_t\big) = \mathrm{I}_{\{T_{i_0-1} < \tau\}} B(t,T_{i_0}). \tag{35}$$

Furthermore, for any $i = i_0+1, \ldots, n$ we have $\mathcal{F}_t \subset \mathcal{F}_{T_{i-1}}$, and thus

$$\hat{U}(t,T_i) = B_t \, \mathbf{E}_{\mathbf{P}^*}\big(B_{T_i}^{-1} \mathrm{I}_{\{T_{i-1} < \tau\}} \,\big|\, \mathcal{F}_t\big) = B_t \, \mathbf{E}_{\mathbf{P}^*}\big(B_{T_{i-1}}^{-1} \mathrm{I}_{\{T_{i-1} < \tau\}} B(T_{i-1},T_i) \,\big|\, \mathcal{F}_t\big).$$

Applying Corollary 2.1, we obtain (as usual, we assume that V does not jump at τ)

$$\hat{U}(t,T_i) = \mathrm{I}_{\{t < \tau\}} \mathbf{E}_{\mathbf{P}^*}\Big\{ \exp\Big(-\int_t^{T_{i-1}} (r_u + \lambda_u)\, du\Big) B(T_{i-1},T_i) \,\Big|\, \mathcal{F}_t \Big\},$$

or equivalently (cf. (32))

$$\hat{U}(t,T_i) = \mathrm{I}_{\{t < \tau\}} \mathbf{E}_{\mathbf{P}^*}\Big\{ \exp\Big(-\int_t^{T_i} (r_u + \lambda_u \mathrm{I}_{[0,T_{i-1}]}(u))\, du\Big) \,\Big|\, \mathcal{F}_t \Big\} = \mathrm{I}_{\{t < \tau\}} B^{\lambda^{i-1}}(t,T_i), \tag{36}$$

where we set $\lambda_t^{i-1} = \lambda_t \mathrm{I}_{[0,T_{i-1}]}(t)$ for $t \in [0,T^*]$. Finally, once again using Corollary 2.1, we get for any $i = i_0, \ldots, n$

$$\tilde{U}(t,T_i) = B_t \, \mathbf{E}_{\mathbf{P}^*}\big(B_{T_i}^{-1} \mathrm{I}_{\{T_i < \tau\}} \big| \mathcal{F}_t\big) = \mathrm{I}_{\{t < \tau\}} \mathbf{E}_{\mathbf{P}^*}\Big\{ \exp\Big(-\int_t^{T_i} (r_u + \lambda_u)\, du\Big) \Big| \mathcal{F}_t \Big\}, \tag{37}$$

so that $\tilde{U}(t,T_i) = \mathrm{I}_{\{t < \tau\}} B^\lambda(t,T_i) = D^0(t,T_i)$. By plugging (35)–(37) into (34), we arrive at the following representation of the price $\tilde{D}^\delta(t,T)$.

Proposition 3.2 *For every $t \leq T$, the price $\tilde{D}^\delta(t,T)$ of a defaultable bond with discrete-time fractional recovery of par equals*

$$\begin{aligned}
\tilde{D}^\delta(t,T) \;\; = \;\; & \mathrm{I}_{\{T_{i_0-1} < \tau\}} \delta B(t,T_{i_0}) + \mathrm{I}_{\{t < \tau\}} \sum_{i=i_0+1}^{n} \delta \, \mathbf{E}_{\mathbf{P}^*}\Big\{ \exp\Big(-\int_t^{T_i} (r_u + \lambda_u^{i-1})\, du\Big) \,\Big|\, \mathcal{F}_t \Big\} \\
& - \mathrm{I}_{\{t < \tau\}} \sum_{i=i_0}^{n} \delta \, \mathbf{E}_{\mathbf{P}^*}\Big\{ \exp\Big(-\int_t^{T_i} (r_u + \lambda_u)\, du\Big) \,\Big|\, \mathcal{F}_t \Big\} \\
& + \mathrm{I}_{\{t < \tau\}} \mathbf{E}_{\mathbf{P}^*}\Big\{ \exp\Big(-\int_t^{T_n} (r_u + \lambda_u)\, du\Big) \,\Big|\, \mathcal{F}_t \Big\},
\end{aligned}$$

where $i_0 = i_0(t) = \inf\{\, i : T_i > t \,\}$ and $\lambda_t^{i-1} = \lambda_t \mathrm{I}_{[0,T_{i-1}]}(t)$. Put another way,

$$\tilde{D}^\delta(t,T) = \mathrm{I}_{\{T_{i_0-1} < \tau\}} \delta B(t,T_{i_0}) + \mathrm{I}_{\{t < \tau\}} \Big(\sum_{i=i_0+1}^{n} \delta \, B^{\lambda^{i-1}}(t,T_i) - \sum_{i=i_0}^{n} \delta \, B^\lambda(t,T_i) + B^\lambda(t,T_n) \Big). \tag{38}$$

Example 3.1 Let us consider a very special case of a T-maturity defaultable bond with a discrete-time recovery, with only two admissible dates $T_0 = 0$ and $T_1 = T$. Since default at time 0 is excluded with probability 1, it is clear that the payment always occurs at time T, no matter whether a bond has defaulted before maturity or not. For any $t \le T$ we have

$$\tilde{D}^\delta(t,T) = B_t \, \mathbf{E}_{\mathbf{P}^*}\big(\delta B_T^{-1} \, \mathrm{I}_{\{0<\tau\le T\}} + B_T^{-1} \, \mathrm{I}_{\{T<\tau\}} \,\big|\, \mathcal{F}_t\big).$$

On the other hand, since $i_0(t) = 1$ for any $t \le T$, formula (38) gives

$$\tilde{D}^\delta(t,T) = \delta B(t,T) + \mathrm{I}_{\{t<\tau\}}(1-\delta) B^\lambda(t,T). \tag{39}$$

Under the present assumptions, since a defaulted bond pays for sure the amount δ at time T, we get $\tilde{D}^\delta(t,T) = \delta B(t,T)$ on the random set $[\tau, T]$, that is, after default. Before default, its value is strictly greater than $\delta B(t,T)$, but we have always $\tilde{D}^\delta(t,T) < B(t,T)$. The last inequality is trivial, since the process λ is strictly positive, and thus $B^\lambda(t,T) < B(t,T)$ for every $t \le T$. We conclude that under the present assumptions, the price of defaultable bond never exceeds the price of the risk-free bond,[1] which is a natural property to require from a model valuing risky debt. On the other hand, for the general model of the continual recovery we have only following equivalence, which holds on the set $\{\tau > t\}$ (of course, $D^\delta(t,T) = 0 < B(t,T)$ on $\{\tau \le t\}$)

$$D^\delta(t,T) \le B(t,T) \quad \Longleftrightarrow \quad \delta \, \mathbf{E}_{\mathbf{P}^*}(B_\tau^{-1} \, \mathrm{I}_{\{t<\tau\le T\}} \,|\, \mathcal{F}_t) \le \mathbf{E}_{\mathbf{P}^*}(B_T^{-1} \, \mathrm{I}_{\{t<\tau\le T\}} \,|\, \mathcal{F}_t).$$

This shows that continual revovery with fractional recovery of par should be handled with care.

3.2 Endogenous Recovery Rules

If Z is not an exogenously given process (but, for instance, a deterministic function of the value process S), the problem of existence and uniqueness of a process S defined by (13) arises. We take the uniqueness of solution to (13) for granted, and we address the problem of pricing of defaultable claims of the form (X, Z, τ), where Z is a specific 'recovery rule,' rather than a given process.

3.2.1 Fractional recovery of market value. Following Duffie and Singleton [22], we assume that $Z_t = (1 - L_t) S_{t-}$, where S is a process we are looking for, and L is a given predictable process. We start with the following lemma, which deals with the process V only. Notice that formula (40) represents a stochastic equation which needs to be solved for the unknown process V.

Lemma 3.1 *Let V satisfy* (19) *with $Z_t = (1 - L_t) V_{t-}$ for some predictable process L, that is,*

$$V_t = \tilde{B}_t \, \mathbf{E}_{\mathbf{P}^*}\Big(\int_t^T \tilde{B}_u^{-1} (1 - L_u) V_u \lambda_u \, du + \tilde{B}_T^{-1} X \,\Big|\, \mathcal{F}_t \Big). \tag{40}$$

Then V is unique, and it is given by the formula

$$V_t = \hat{B}_t \, \mathbf{E}_{\mathbf{P}^*}\big(\hat{B}_T^{-1} X \,\big|\, \mathcal{F}_t\big) \tag{41}$$

with

$$\hat{B}_t = \exp\Big(\int_0^t (r_u + \lambda_u L_u) \, du \Big). \tag{42}$$

Proof. In view of (24) we have

$$dV_t = V_t(r_t + \lambda_t) \, dt - (1 - L_t) V_t \lambda_t \, dt + \tilde{B}_t \, dN_t,$$

or equivalently,

$$dV_t = V_t(r_t + \lambda_t L_t) \, dt + \tilde{B}_t \, dN_t.$$

---

[1] This holds true also in the case of zero recovery considered in Section 3.1.1.

This immediately yields (41) (as usual, we assume that the last term follows a martingale). Of course, this proves also that equation (40) admits a unique solution. □

The next step is to examine the relationship between V (or rather U) and the price process of a defaultable claim. In view of Proposition 2.1 (which we may apply since the process $Z_t = (1-L_t)V_{t-}$ is predictable), we find that the process $U_t = \mathrm{I}_{\{t<\tau\}} V_t$ satisfies

$$U_t = B_t \, \mathbf{E}_{\mathbf{P}^*} \Big(B_\tau^{-1} \big((1-L_\tau)V_{\tau-} + \Delta V_\tau \big) \, \mathrm{I}_{\{t<\tau\le T\}} + B_T^{-1} X \, \mathrm{I}_{\{T<\tau\}} \, \Big| \, \mathcal{F}_t \Big). \tag{43}$$

Corollary 3.1 *Let the process V be given by formula* (40) *for some predictable process L. Assume that $\Delta V_\tau = 0$. Then the process $U_t = \mathrm{I}_{\{t<\tau\}} V_t$ satisfies*

$$U_t = \mathrm{I}_{\{t<\tau\}} \hat{B}_t \, \mathbf{E}_{\mathbf{P}^*} \big(\hat{B}_T^{-1} X \, \big| \, \mathcal{F}_t \big) \tag{44}$$

and

$$U_t = B_t \, \mathbf{E}_{\mathbf{P}^*} \Big(B_\tau^{-1} (1-L_\tau) U_{\tau-} \, \mathrm{I}_{\{t<\tau\le T\}} + B_T^{-1} X \, \mathrm{I}_{\{T<\tau\}} \, \Big| \, \mathcal{F}_t \Big). \tag{45}$$

Proof. Equality (44) is an immediate consequence of (41). The second formula follows from (43) (we use the trivial equality $U_{\tau-} = V_{\tau-}$). □

In view of Corollary 3.1, the process U satisfies equation (45), that is, the implicit definition of the price process S. Note that we have not proved that the uniqueness of solutions holds for the equation

$$S_t = B_t \, \mathbf{E}_{\mathbf{P}^*} \Big(B_\tau^{-1} (1-L_\tau) S_{\tau-} \, \mathrm{I}_{\{t<\tau\le T\}} + B_T^{-1} X \, \mathrm{I}_{\{T<\tau\}} \, \Big| \, \mathcal{F}_t \Big). \tag{46}$$

We have merely shown that (46) admits a solution. The uniqueness of solutions to (46) can be deduced from standard results on backward SDEs, however. To this end, it might be convenient to use the equivalent representation of equation (46), namely (cf. (17))

$$S_t = \mathbf{E}_{\mathbf{P}^*} \Big(\int_t^T S_u \big((1-L_u)h_u - r_u \big) \, du + X \, \mathrm{I}_{\{T<\tau\}} \, \Big| \, \mathcal{F}_t \Big). \tag{47}$$

3.2.2 General recovery rule. In principle, we may also deal with a 'general recovery rule', more precisely, we may assume that the payoff process Z satisfies $Z_t = p(t, S_{t-})$, where the function $p(t,s)$ is Lipschitz continuous with respect to s, and satisfies $p(t,0) = 0$. In this case, however, we have merely the following result, which again is a consequence of Proposition 2.1 (once again, the problem of existence and uniqueness of solutions to (49) and (51) is not addressed here; this follows from standard results on backward SDEs).

Corollary 3.2 *Let S be the unique solution to the backward SDE*

$$S_t = B_t \, \mathbf{E}_{\mathbf{P}^*} \Big(B_\tau^{-1} p(\tau, S_{\tau-}) \, \mathrm{I}_{\{t<\tau\le T\}} + B_T^{-1} X \, \mathrm{I}_{\{T<\tau\}} \, \Big| \, \mathcal{F}_t \Big), \tag{48}$$

or equivalently, to the equation (*cf.* (17))

$$S_t = \mathbf{E}_{\mathbf{P}^*} \Big(\int_t^T \big(p(u, S_u) h_u - r_u S_u \big) \, du + X \, \mathrm{I}_{\{T<\tau\}} \, \Big| \, \mathcal{F}_t \Big). \tag{49}$$

Let V be the unique solution to the backward SDE

$$V_t = \tilde{B}_t \, \mathbf{E}_{\mathbf{P}^*} \Big(\int_t^T \tilde{B}_u^{-1} p(u, V_u) \lambda_u \, du + \tilde{B}_T^{-1} X \, \Big| \, \mathcal{F}_t \Big), \tag{50}$$

or equivalently, to the equation

$$V_t = \mathbf{E}_{\mathbf{P}^*} \Big(\int_t^T \big(p(u, V_u)\lambda_u - (r_u + \lambda_u) V_u \big) \, du + X \, \Big| \, \mathcal{F}_t \Big). \tag{51}$$

If $\Delta V_\tau = 0$, then $S_t = \mathrm{I}_{\{t<\tau\}} V_t$. Otherwise, S is given by formula (27).

4 Credit Ratings

To produce a tractable model which accounts for the migration between of rating classes, Jarrow et al. [32] make the following, rather stringent, assumptions: (i) there exists a unique equivalent martingale measure $\mathbf{P}^*$ making all default-free and default-risky zero-coupon bond prices martingales, after normalization by the savings account, (ii) the default time τ is independent of the risk-free rate r under the martingale measure $\mathbf{P}^*$, (iii) the recovery coefficient is a constant δ.

They first develop a discrete-time model which takes account for the migration of a defaultable bond in the finite set of *credit rating classes.* Subsequently, a continuous-time counterpart is also examined. Methodology developed in Jarrow et al. [32] is a direct extension of the approach in Jarrow and Turnbull [34]. They assume that a defaulted bond pays at maturity a fraction of its par value.[2] Therefore, the price at time $t \le T$ of a T-maturity defaultable bond equals

$$\tilde{D}^{\delta}(t,T) = \mathbf{E}_{\mathbf{P}^*}\Big(\exp\Big(-\int_t^T r_u\,du\Big)\big(\delta\,\mathrm{I}_{\{T\ge\tau\}} + \mathrm{I}_{\{T<\tau\}}\big)\,\Big|\,\mathcal{F}_t\Big), \tag{52}$$

where τ is the default time, and δ is the constant recovery rate. Suppose that we have chosen a model for the short-term rate r. It is clear from expression (52) that we need only to model a stopping time τ. In addition, under assumption (i), formula (52) can be substantially simplified, namely,

$$\tilde{D}^{\delta}(t,T) = B(t,T)\,\mathbf{E}_{\mathbf{P}^*}\big(\delta\,\mathrm{I}_{\{T\ge\tau\}} + \mathrm{I}_{\{T<\tau\}}\,\big|\,\mathcal{F}_t\big). \tag{53}$$

Consequently (it might be instructive to compare (54) with (39)),

$$\tilde{D}^{\delta}(t,T) = B(t,T)\,\big(\delta + (1-\delta)\mathbf{P}^*\{T<\tau\,|\,\mathcal{F}_t\}\big) = B(t,T)\,\big(1-(1-\delta)\mathbf{P}^*\{T\ge\tau\,|\,\mathcal{F}_t\}\big). \tag{54}$$

As will soon become clear, the stopping time τ is explicitly dependent on the initial rating of a particular bond. Therefore, expressions (52)–(54) should be seen as generic valuation formulae for defaultable bonds.

Given an initial rating of a defaultable bond, the future changes in its assessments by a rating agency are described by a stochastic process, referred to as the *migration process.* Formally, for a given bond, the value at time t of the associated migration process coincides with its current rating. There is no loss of generality, if we assume that the set of rating classes is $\{1,\dots,K\}$, where the state K is assumed to correspond to the default event. It is assumed that the migration process, C say, follows a Markov chain (under both real-world probability $\mathbf{P}$ and the spot martingale measure $\mathbf{P}^*$), that is, the future evolution of ratings classes a particular bond does not depend on the bond's history, but only on its current rating.

4.1 Discrete-time Model

In a discrete-time setup, the migration process and the default time are assumed to satisfy: (iv) the migration process C follows, under the real-world probability $\mathbf{P}$, a time-homogeneous Markov chain with the transition matrix (by definition, $p_{ij} = \mathbf{P}\{C_{t+1} = j\,|\,C_t = i\}$)

$$P = [p_{ij}]_{1\le i,j\le K}, \quad p_{i,j} \ge 0, \quad \sum_{j=1}^{K} p_{ij} = 1,$$

with $p_{Kj} = 0$ for every $j < K$ (so that $p_{KK} = 1$ for every t, that is, the state K is absorbing), and (v) C follows a (time-inhomogeneous) Markov chain under $\mathbf{P}^*$, with time-dependent transition matrix

$$Q(t) = [q_{ij}(t,t+1)]_{1\le i,j\le K}, \quad q_{ij}(t,t+1) \ge 0, \quad \sum_{j=1}^{K} q_{ij}(t,t+1) = 1,$$

with $q_{Kj}(t,t+1) = 0$ for every $j < K$ and t (so that once again the state K is absorbing).

---

[2] This convention coincides with the concept of discrete-time fractional recovery of par introduced in Section 3, provided that we take $T_0 = 0$ and $T_1 = T$ (cf. Example 3.1).

The default time τ is the first moment the rating process hits the state K (the horizon date T^* is assumed to be a natural number, and the infimum over an empty set equals, by convention, $+\infty$)

$$\tau := \inf \{ t \in \{0, 1, \ldots, T^*\} \, : \, C_t = K \}. \tag{55}$$

To ensure analytical tractability of the model, an additional 'technical' assumption is made. Namely, it is postulated that the following relationship holds

$$q_{ij}(t, t+1) = \pi_i(t) p_{ij}, \quad \forall \, i \neq j, \tag{56}$$

where time-dependent coefficients $\pi_i(t)$ are interpreted as discrete-time *risk premia.* The last assumption implies, in particular, that

$$q_{ii}(t, t+1) = 1 + \pi_i(t)(p_{ii} - 1), \quad \forall \, i.$$

In other words, for any state i, the probability under the martingale measure $\mathbf{P}^*$ of jumping to the state $j \neq i$ is assumed to be proportional to the corresponding probability under the real-world probability $\mathbf{P}$, with the proportionality factor which may depend on i and t, but not on j.

Assume that we are given the initial term structures of default-free and defaultable bonds, and the real-world transition matrix P (in principle, all these quantities can be 'observed'). Then, under the above set of assumptions, Jarrow et al. [32] offer a recursive procedure which leads to the unique determination of the 'risk premium' process $\pi(t)$, $t = 0, \ldots, T-1$. Consequently, the time-dependent transition matrix $Q(t)$ under $\mathbf{P}^*$ is also uniquely specified.

4.2 Continuous-time Model

A similar approach is developed in the continuous-time setup. It is postulated that: (iv') under the real-world probability $\mathbf{P}$, the migration process C follows a time-homogeneous Markov chain, with intensity matrix $\tilde{\Lambda}$ satisfying mild 'technical' conditions (which guarantee that the state K is absorbing, and a suitable monotonicity of default probabilities holds), (v') under the martingale measure $\mathbf{P}^*$, the migration process also follows a Markov chain, but with a possibly time-dependent intensity matrix $\Lambda(t)$. As before, the default time τ is the first time the rating process hits the absorbing state K.

Tractability condition (56) now takes the following form: there exists a diagonal matrix U, whose first $K-1$ entries, $U_{ii}(t)$, $i = 1, \ldots, K-1$, are strictly positive deterministic functions, and the last entry, $U_{KK}(t) = 1$ for every t, such that risk-neutral and real-world intensity matrices satisfy

$$\Lambda(t) = U(t) \tilde{\Lambda}, \quad \forall \, t \in [0, T^*]. \tag{57}$$

Suppose that the initial term structures of default-free and default-risky zero-coupon bonds are known. Then for any choice of the 'historical' intensity matrix $\tilde{\Lambda}$, one can produce a model for defaultable term structure in two steps. In the first step, we construct the migration process C under the real-world probability $\mathbf{P}$, using the intensity matrix $\tilde{\Lambda}$ (by assumption, the migration process is independent of the underlying risk-free short-term rate r). Subsequently, we search for an equivalent probability measure $\mathbf{P}^*$, which would reproduce the observed prices of all defaultable bonds through the risk-neutral valuation formula (54). If we denote by $\tilde{D}^{\delta}_i(0, T)$ the initial price of the defaultable bond which belongs to the i^{th} rating class at time 0, then we have

$$\tilde{D}^{\delta}_i(0, T) = B(0, T) \big(\delta + (1 - \delta) \mathbf{P}^* \{ T < \tau \,|\, C_0 = i \} \big). \tag{58}$$

Since τ is given by (55) and the state K absorbing, it is also clear that

$$\mathbf{P}^* \{ T < \tau \,|\, C_0 = i \} = \mathbf{P}^* \{ C_T = K \,|\, C_0 = i \} = q_{iK}(0, T),$$

where $Q(0, T) = [q_{ij}(0, T)]_{\, 1 \leq i, j \leq K}$ is the transition matrix corresponding to the time interval $[0, T]$.

5 State Variables

In this section – in which we follow Duffie and Singleton [22] and Lando [41] – we place ourselves again within the general framework, as presented in Section 2. In order to make the model of Section 2 analytically more tractable, we impose additional conditions on the default time τ – more specifically, on the intensity process h of the default process N. It should be stressed that additional conditions of this kind are complementary to those presented in Section 4. For instance, it seems natural to consider a model of defaultable debt which combines the presence of the migration process C with the presence of the state variables process Y (as in Lando [41]).

We assume that we are given a k-dimensional stochastic process Y defined on the underlying probability space $(\Omega, \mathcal{F}, \mathbf{P}^*)$. The process Y, which typically is assumed to be Markovian under the spot martingale measure $\mathbf{P}^*$, is assumed to model the dynamics of 'state variables' which underpin the evolution of all other variables in our model of the economy. As far as the default time is concerned, we postulate that τ is the first jump time of the Cox process N with the intensity of the form $\lambda_t = \lambda(Y_t)$, for some function $\lambda : \mathbf{R}^k \to \mathbf{R}_+$. At this stage, no explicit distinction between defaultable bonds with different rating assessments is made. In other words, we focus on a bond which currently belongs to a particular class, and we exclude the possibility of migration to any other class then to the 'default class' (this restriction will be relaxed in Section 5.1).

The construction of the default time τ with these properties can be achieved as follows. Let $(\mathcal{G}_t)_{\,t\in[0,T^*]}$ be the filtration generated by the process Y, and let η be a random variable independent of Y (of course, η is also assumed to be defined on $(\Omega, \mathcal{F}, \mathbf{P}^*)$, so that a suitable enlargement of the underlying probability space might be required). More specifically, we assume that η has a unit exponential probability law under $\mathbf{P}^*$. To define default time τ (that is, the first jump of the Cox process N), we set

$$\tau = \inf \Big\{ t \in \mathbf{R}_+ \;\Big|\; \int_0^t \lambda(Y_u)\, du \geq \eta \Big\}. \tag{59}$$

To make such a specification of the default time τ useful, we need to assume, in addition, that X is a $\mathcal{G}_T$-measurable random variable, the recovery process Z is predictable with respect to the filtration $(\mathcal{G}_t)_{\,t\in[0,T^*]}$, and, for instance, $r_t = r(Y_t)$ (this agrees with our interpretation of Y as state-variables process). Under this set of assumptions, in all previously established formulae in which the default time τ does not appear explicitly, that is, the presence of the default process N is manifested only through its intensity process $\lambda_t = h(Y_t)$, we may substitute the conditional expectation with respect to $\mathcal{F}_t$ with the conditional expectation with respect to $\mathcal{G}_t$. For instance, using Corollary 2.1 with $\Delta V_\tau = 0$, we obtain

$$S_t = \mathrm{I}_{\{t<\tau\}} \mathbf{E}_{\mathbf{P}^*} \Big(\int_t^T \exp \Big(-\int_t^u R(Y_v)\, dv \Big) Z_u \lambda(Y_u)\, du + \exp \Big(-\int_t^T R(Y_v)\, dv \Big) X \,\Big|\, \mathcal{G}_t \Big), \tag{60}$$

where $R(Y_u) = r(Y_u) + h(Y_u)$. Formula (60) is a direct consequence of equality (28), combined with the observation that

$$\mathcal{G}_t \subset \mathcal{F}_t \subset \mathcal{G}_t \vee \sigma(\eta), \tag{61}$$

where, by assumption, the σ-fields $\mathcal{G}_T$ and $\sigma(\eta)$ are mutually independent. As shown by Lando [41], in the present framework formula (60) can be derived in a more straightforward way, without making explicit reference to the pre-default value process V (that is, using Lemma 2.1 rather than Corollary 2.1).

Proposition 5.1 *Let the default time τ be given by (59). Then we have*

$$S_t = \mathrm{I}_{\{t<\tau\}} \tilde{B}_t\, \mathbf{E}_{\mathbf{P}^*} \Big(\int_t^T \tilde{B}_u^{-1} Z_u \lambda(Y_u)\, du + \tilde{B}_T^{-1} X \,\Big|\, \mathcal{G}_t \Big), \tag{62}$$

where the process $\tilde{B}$ is given by (20) with $r_u = r(Y_u)$ and $\lambda_u = h(Y_u)$.

Proof. Notice that for any $0 \le t \le u$ we have

$$\mathbf{P}^*\{\tau > u \,|\, \mathcal{G}_T \vee \mathcal{H}_t\} = \begin{cases} \exp\big(-\int_t^u \lambda(Y_v)\,dv\big), & \text{on the set } \{\tau > t\}, \\ 0, & \text{otherwise,} \end{cases}$$

where $\mathcal{H}_t = \sigma(H_u : u \le t)$ (for the definition of H, see Assumption (A.2)). Therefore (cf. (16)),

$$\begin{aligned} S_t &= B_t\,\mathbf{E}_{\mathbf{P}^*}\Big(\int_t^T B_u^{-1} Z_u \lambda(Y_u)\,\mathrm{I}_{\{u \le \tau\}}\,du + B_T^{-1} X\,\mathrm{I}_{\{T<\tau\}} \,\Big|\, \mathcal{F}_t\Big) \\ &= B_t\,\mathbf{E}_{\mathbf{P}^*}\Big(\int_t^T B_u^{-1} Z_u \lambda(Y_u)\,\mathbf{P}^*\{\tau \ge u \,|\, \mathcal{G}_T \vee \mathcal{H}_t\}\,du + B_T^{-1} X\,\mathbf{P}^*\{\tau > T \,|\, \mathcal{G}_T \vee \mathcal{H}_t\} \,\Big|\, \mathcal{F}_t\Big) \\ &= B_t\,\mathbf{E}_{\mathbf{P}^*}\Big(\int_t^T B_u^{-1} Z_u \lambda(Y_u)\exp\Big(-\int_t^u \lambda(Y_v)\,dv\Big)\,du + B_T^{-1} X \exp\Big(-\int_t^T \lambda(Y_v)\,du\Big) \,\Big|\, \mathcal{F}_t\Big) \\ &= \tilde{B}_t\,\mathbf{E}_{\mathbf{P}^*}\Big(\int_t^T \tilde{B}_u^{-1} Z_u \lambda(Y_u)\,du + \tilde{B}_T^{-1} X \,\Big|\, \mathcal{F}_t\Big) \end{aligned}$$

on the set $\{\tau > t\}$ ($= 0$ otherwise). We wish now to replace $\mathcal{F}_t$ with $\mathcal{G}_t$ in the last expression. In view of (61), it it enough to observe that conditioning with respect to $\mathcal{G}_t$ coincides in our case with conditioning with respect to $\mathcal{G}_t \vee \sigma(\eta)$. This follows immediately from the fact that the random variable η is independent of the process Y, and the random variable under the sign of conditional expectation is measurable with respect to the σ-field $\mathcal{G}_T$. □

Proposition 5.1, when combined with Corollary 2.1, seems to suggest that the jump ΔV_τ, even if it does not vanish, plays no longer an important role in the present setup. Indeed, it shows that we always have $S_t = \mathrm{I}_{\{t<\tau\}} V_t$, where the process V is given by (19). Consequently, combining (14) with (21), we find that under the present assumptions the pre-default process associated with any defaultable claim (X, Z, τ) satisfies

$$\mathbf{E}_{\mathbf{P}^*}\big(B_\tau^{-1} \Delta V_\tau\,\mathrm{I}_{\{t<\tau\le T\}} \,\big|\, \mathcal{F}_t\big) = 0, \quad \forall\, t \in [0,T].$$

Example 5.1 The paper by Duffie and Singleton [22] focuses on the special case of fractional recovery of market value. They assume that: (i) there is a state-variables process Y that is Markovian under the spot martingale measure $\mathbf{P}^*$, (ii) the promised contingent claim is of the form $X = g(Y_T)$ for some function $g : \mathbf{R}^k \to \mathbf{R}$, (iii) the default-adjusted short-term rate $R_t = r_t + \lambda_t L_t = \rho(Y_t)$ for some function $\rho : \mathbf{R}^k \to \mathbf{R}$. Under (i)–(iii), we have

$$V_t = \mathbf{E}_{\mathbf{P}^*}\Big\{\exp\Big(-\int_t^T \rho(Y_u)\,du\Big) g(Y_T) \,\Big|\, Y_t\Big\}. \tag{63}$$

Moreover, if Y follows a non-degenerate diffusion process, then $\Delta V_\tau = 0$ and thus $S_t = \mathrm{I}_{\{t<\tau\}} V_t$. Indeed, in this case the martingale given by formula (23) – in which once again we may and do substitute $\mathcal{F}_t$ with $\mathcal{G}_t$ – is continuous. Consequently, in view of (22), the process V is also continuous.

5.1 Conditionally Markov Ratings Process

We shall now describe an extension – due to Lando [41] – of the credit ratings model elaborated by Jarrow et al. [32] (the latter model was discussed in Section 4). As usual, we assume that the spot martingale measure $\mathbf{P}^*$, and risk-free term structure $B(t,T)$ are given. Lando [41] modifies the approach in [32] by introducing a conditionally Markov migration process, which is due to account accounts for both the presence of different rating classes, and for the postulated existence of the underlying state variables, as modeled by a process Y. It appears that this can be achieved by a suitable modification of the migration process C introduced in Section 4 (whenever possible, the notation introduced in Section 4 is preserved).

We place ourselves in a continuous-time setup. The migration process C is now assumed to follow, under the spot martingale measure, a conditional Markov chain with the (stochastic) intensity matrix

$$\Lambda(Y_t) = [\lambda_{ij}(Y_t)]_{1\le i,j\le K}, \quad \forall\, t \in [0,T^*],$$

which is assumed to satisfy, for every $t \in [0,T^*]$,

$$\lambda_{ii}(Y_t) = -\sum_{j=1,\, j\neq i}^{K} \lambda_{ij}(Y_t), \quad \text{and} \quad \lambda_{K,i}(Y_t) = 0, \quad \forall\, i = 1,\dots,K, \tag{64}$$

where $\lambda_{ij} : \mathbf{R}^k \to \mathbf{R}_+$ are non-negative functions. For any such a matrix, given the process Y and the initial rating i (at time 0, say), it is possible to construct a migration process C corresponding to the matrix $\Lambda(Y_t)$. More specifically, the migration process C is assumed to follow, conditionally on the path of the state-variables process Y, a Markov chain with finite state space $\{1,\dots,K\}$ and time-dependent (but deterministic) intensity matrix $\Lambda(Y_t)$. It follows from (64) that the K^{th} row of the matrix $\Lambda(Y_t)$ is assumed to vanish indentically, so that K is an absorbing state. As in Section 4, the absorbing state K represents the default event, since the default time is the first time the migration process C hits K. The construction of a process C with these properties is a straightforward generalization of the construction provided by formula (59) (though we need to deal with an infinite family of mutually independent exponentially distributed random variables).

Due to the nature of the default time τ, the valuation of defaultable claims becomes more cumbersome. In fact, since default time τ and short-term rate r are no longer mutually independent (as was postulated in Jarrow et al. [32]), no shortcuts, such as formula (54), are available, Therefore, one needs to use the standard definition (14) of the price of a defaultable claim. This applies even to the case of a zeru-coupon bond, under the assumption that the recovery rate equals 0 (that is, when the recovery process Z vanishes identically). By definition, the price of such a bond equals (cf. (14) or (33))

$$D_i^0(t,T) = B_t\, \mathbf{E}_{\mathbf{P}^*}\big(B_T^{-1}\, \mathrm{I}_{\{T<\tau\}} \,\big|\, \mathcal{F}_t \vee \{C_t = i\}\big),$$

where we assume that at time t the bond belongs to the i^{th} rating class, for some $i < K$. Using a similar reasoning as in the proof of Proposition 5.1 (that is, conditioning first on the future evolution of the process Y), we find that

$$D_i^0(t,T) = B_t\, \mathbf{E}_{\mathbf{P}^*}\big(B_T^{-1}(1 - p_{iK}^Y(t,T)) \,\big|\, \mathcal{G}_t\big), \tag{65}$$

where

$$p_{iK}^Y(t,T) = \mathbf{P}^*\big\{C_T = K \,|\, \{C_t = i\} \vee \sigma(Y_u \,:\, u \in [t,T])\big\}. \tag{66}$$

Notice that $p_{iK}^Y(t,T)$ is simply the conditional transition probability of the migration process C, over the time interval $[t,T]$, with conditioning on the future behaviour of the state-variables process Y. Evaluation of $p_{iK}^Y(t,T)$, given a particular sample path of the process Y, would be thus a relatively simple task, were the intensity matrix $\Lambda(Y_t)$ a diagonal matrix. Indeed, we would be then able to separate variables in the corresponding system of Kolmogorov differential equations. A similar – but slightly less explicit – result holds provided that

$$\Lambda(Y_t) = B\,\Gamma(Y_t)B^{-1},$$

where $\Gamma(Y_t)$ is a diagonal matrix, and B is a $K \times K$ matrix whose columns are the eigenvectors of $\Lambda(Y_t)$. Under this rather restrictive condition, Lando [41] derived a quasi-explicit valuation formula for a defaultable bond, and indeed for any (promised) European claim of the form $X = g(Y_T, C_T)$.

To conclude, the problem of valuation of defaultable debt is reduced to finding of a convenient representation of the right-hand side in (66), which would subsequently allow to evaluate the conditional expectation in (65). Generally speaking, this seems to be a rather difficult task, especially when restrictive regularity conditions are not imposed on the intensity matrix, or when we deal with a non-zero recovery rate. In any case, valuation of defaultable claims can be done through simulation techniques though.

6 Credit spreads

The exposition in this section is motivated by results of Schönbucher [55]–[56] (see also Duffie [16]), though the line of argument is somewhat different. Generally speaking, the main modification is that we do not assume here that the default time of a T-maturity defaultable bond is prespecified. We postulate only that we are given a defaultable term structure, represented by a family of defaultable instantaneous forward rates. Our goal is thus to explain exogenously given defaultable term structure through an introduction of a family of default times (which are defined on a suitable enlargement of the underlying probability space). It should be stressed that in this section we do not price defaultable bonds for a given risk-free term structure and a prespecified default time, as was the case, for instance, in Sections 2–3. On the contrary, we assume that the 'pre-default' values of defaultable bonds are given a priori, and we search for an arbitrage free setup supporting these values. The reason for chosing such an approach is that we feel that a simultaneous specification of both the default times (or equivalently, intensities of default) and the defaultable forward rates would lead to an overspecified model of defaultable term structure. Notice, in addition, that if one postulates that all bonds of a given rating class have identical intensity of default[3] – the maturity of a particular bond notwithstanding – one looses a degree of flexibility.

6.1 Zero Recovery

In the first step, we ficus on a particular defaultable bond so that the maturity date T is fixed. In this case, it is enough to assume that the dynamics of a defaultable instantaneous forward rate for the date T is specified. We wish to explain this dynamics by introducing a judiciously chosen stopping time, interpreted as default time with zero recovery. Subsequently, we address the following natural question: under which conditions, a default time τ can be chosen simultaneuosly for all defaultable bonds. We make the following standing assumptions.

(B.1) We are given a d-dimensional standard Brownian motion W, defined on the underlying real-world probability space $(\Omega, (\mathcal{F}_t)_{t\in[0,T^*]}, \mathbf{P})$.

(B.2) For any fixed maturity $T \leq T^*$, the risk-free instantaneous forward rate $f(t,T)$ satisfies[4]

$$df(t,T) = \alpha(t,T)\,dt + \sigma(t,T)\cdot dW_t, \tag{67}$$

where α and σ are adapted stochastic processes with values in $\mathbf{R}$ and $\mathbf{R}^d$, respectively.

(B.3) The defaultable instantaneous forward rate $g(t,T)$ satisfies

$$dg(t,T) = \tilde{\alpha}(t,T)\,dt + \tilde{\sigma}(t,T)\cdot dW_t, \tag{68}$$

for some processes $\tilde{\alpha}$ and $\tilde{\sigma}$.

Assumptions (B.1)–(B.2) are, of course, the standard asssumptions of the Heath-Jarrow-Morton [29] approach to term structure modeling. By definition, the price of a T-maturity default-free zero-coupon bond equals

$$B(t,T) := \exp\Big(-\int_t^T f(t,u)\,du\Big), \quad \forall\, t \in [0,T]. \tag{69}$$

The meaning of Assumption (B.3) is not as transparent yet. We set, however,

$$\tilde{D}^0(t,T) := \exp\Big(-\int_t^T g(t,u)\,du\Big), \quad \forall\, t \in [0,T], \tag{70}$$

---

[3] This seems to be a standard assumption in all existing rating-based models.

[4] For technical conditions under which formulae (67)–(68) make sense, see Heath et al. [29].

and we interpret $\tilde{D}^0(t,T)$ as the pre-default value of a T-maturity defaultable zero-coupon bond. Of course, for this interpretation to be justified, we need first to develop an arbitrage-free model for default-free and defaultable term structures. In particular, our goal is to show that the pre-default value $\tilde{D}^0(t,T)$ can be indeed be interpreted as the price before default of a T-maturity defaultable zero-coupon bond with zero recovery rate, in an arbitrage-free framework. To this end, we shall assume, in addition, that the *credit spread* $g(t,T) - f(t,T)$ is strictly positive, so that $\tilde{D}^0(t,T) < B(t,T)$. For the reader's convenience, we quote the following result (see [29]).

Lemma 6.1 *The bond price $B(t,T)$ satisfies*

$$dB(t,T) = B(t,T)\big(a(t,T)\,dt + b(t,T)\cdot dW_t\big), \tag{71}$$

where

$$a(t,T) = f(t,t) - \alpha^*(t,T) + \tfrac{1}{2}\,|b(t,T)|^2, \quad b(t,T) = -\sigma^*(t,T),$$

and

$$\alpha^*(t,T) = \int_t^T \alpha(t,u)\,du, \quad \sigma^*(t,T) = \int_t^T \sigma(t,u)\,du.$$

An analogous result holds for $\tilde{D}^0(t,T)$, with an obvious change of notation, so that

$$d\tilde{D}^0(t,T) = \tilde{D}^0(t,T)\big(\tilde{a}(t,T)\,dt + \tilde{b}(t,T)\cdot dW_t\big) \tag{72}$$

with

$$\tilde{a}(t,T) = g(t,t) - \tilde{\alpha}^*(t,T) + \tfrac{1}{2}\,|\tilde{b}(t,T)|^2, \quad \tilde{b}(t,T) = -\tilde{\sigma}^*(t,T). \tag{73}$$

Assume now, in addition, that one may also invest in the risk-free savings account B, which corresponds to the short-term rate $r_t = f(t,t)$. In view of (71), the relative price $Z(t,T) = B_t^{-1}B(t,T)$ satisfies under $\mathbf{P}$

$$dZ(t,T) = -Z(t,T)\Big(\big(\alpha^*(t,T) - \tfrac{1}{2}|b(t,T)|^2\big)\,dt - b(t,T)\cdot dW_t\Big).$$

The following condition is known to exclude arbitrage across all default-free bonds, as well as between default-free bonds and the savings account.

Condition (M) There exists an adapted $\mathbf{R}^d$-valued process γ such that

$$\mathbf{E_P}\Big\{\exp\Big(\int_0^{T^*}\gamma_u\cdot dW_u - \frac{1}{2}\int_0^{T^*}|\gamma_u|^2\,du\Big)\Big\} = 1$$

and, for any maturity $T \le T^*$, we have

$$\alpha^*(t,T) = \tfrac{1}{2}|\sigma^*(t,T)|^2 - \sigma^*(t,T)\cdot\gamma_t.$$

Let γ be some process satisfying Condition (M). Then the probability measure $\mathbf{P}^*$, given by the formula

$$\frac{d\mathbf{P}^*}{d\mathbf{P}} = \exp\Big(\int_0^{T^*}\gamma_u\cdot dW_u - \frac{1}{2}\int_0^{T^*}|\gamma_u|^2\,du\Big), \quad \mathbf{P}\text{-a.s.},$$

is a spot martingale measure for the default-free term structure. Moreover, if we define a Brownian motion W^* under $\mathbf{P}^*$ by setting

$$W_t^* = W_t - \int_0^t \gamma_u\,du, \quad \forall\, t \in [0,T^*],$$

then, for any fixed maturity $T \le T^*$, the default-free bond price $B(t,T)$ satisfies under $\mathbf{P}^*$

$$dB(t,T) = B(t,T)\big(r_t\,dt + b(t,T)\cdot dW_t^*\big). \tag{74}$$

We shall assume from now on that the process γ is uniquely determined, so that the default-free bonds market is complete.[5] Formally, this means that any default-free contingent claim can be priced through risk-neutral valuation formula. It should be stressed, however, that this remark does not apply to defaultable claims. After a recollection of the well-known facts about the Heath-Jarrow-Morton approach, we shall now focus on dynamics of the relative pre-default value of a defaultable bond. First, under $\mathbf{P}$ the process $\tilde{Z}(t,T) = B_t^{-1}\tilde{D}^0(t,T)$ satisfies

$$d\tilde{Z}(t,T) = \tilde{Z}(t,T)\big((\tilde{a}(t,T) - f(t,t))\,dt + \tilde{b}(t,T)\cdot dW_t\big). \tag{75}$$

Consequently, under the unique spot martingale measure $\mathbf{P}^*$, we have

$$d\tilde{Z}(t,T) = \tilde{Z}(t,T)\big(\lambda_t\,dt + \tilde{b}(t,T)\cdot dW_t^*\big), \tag{76}$$

where we set (notice that the process λ may depend on maturity T, in general)

$$\lambda_t := \tilde{a}(t,T) - f(t,t) + \tilde{b}(t,T)\gamma_t. \tag{77}$$

The no-arbitrage condition between a defaultable bond and savings account reads:[6] $\lambda_t = 0$ for $t \le T$. It is easily seen that this condition is never satisfied, under the present assumptions. Indeed, were it true, $\tilde{Z}(t,T)$ would be a $\mathbf{P}^*$-martingale, and we would have

$$\tilde{D}^0(t,T) = \mathbf{E}_{\mathbf{P}^*}\Big\{\exp\Big(-\int_t^T f(u,u)\,du\Big)\,\Big|\,\mathcal{F}_t\Big\} = B(t,T), \quad \forall\, t\in[0,T].$$

This would contradict our assumption that $\tilde{D}^0(t,T) < B(t,T)$. Therefore, we shall assume from now on that the process λ does not vanish identically, for any maturity T. From the property that credit spread $g(t,u) - f(t,u)$ is strictly positive, we should be able to deduce that λ is strictly positive.[7]

6.1.1 Defaultable bond. Let us fix a maturity date T. In view of (76) we have

$$d\tilde{Z}(t,T) = \tilde{Z}(t,T)\big(\lambda_t\,dt + \tilde{b}(t,T)\cdot dW_t^*\big).$$

We shall show that there exists a stopping time τ, such that the process (as before, $H_t = \mathrm{I}_{\{t\ge\tau\}}$)

$$M_t = H_t - \int_0^t \lambda_u\,\mathrm{I}_{\{u\le\tau\}}\,du \tag{78}$$

follows a local martingale under $\mathbf{P}^*$ (or rather, under a suitable extension $\mathbf{Q}^*$ of $\mathbf{P}^*$, which we are now going to introduce). The existence of τ follows easily from standard results in the theory of stochastic processes, provided that we allow for a suitable enlargement of the underlying probability space. In fact, we cannot expect a stopping time τ with desired properties to exist on the original probability space $(\Omega, (\mathcal{F}_t)_{t\in[0,T^*]}, \mathbf{P}^*)$, in general. For instance, if the underlying filtration is generated by a standard Brownian motion, which is the usual assumption imposed to ensure the uniqueness of the spot martingale measure $\mathbf{P}^*$, no stopping time with desired properties exists on the original space. Let us denote by $(\tilde{\Omega}, (\tilde{\mathcal{F}}_t)_{t\in[0,T^*]}, \mathbf{Q}^*)$ the enlarged probability space. Our additional requirement is that W^* remains a standard Brownian motion when we switch from $\mathbf{P}^*$ to $\mathbf{Q}^*$. To satisfy all these requirements, it is enough to take a product space $(\Omega\times\hat{\Omega}, (\tilde{\mathcal{F}}_t\otimes\hat{\mathcal{F}})_{t\in[0,T^*]}, \mathbf{P}^*\otimes\mathbf{Q})$ where the probability $(\hat{\Omega}, \hat{\mathcal{F}}, \mathbf{Q})$ is large enough to support a unit exponential random variable, η say. Then we may put (cf. (59))

$$\tau = \inf\Big\{t\in\mathbf{R}_+ \;\Big|\; \int_0^t \lambda_u\,du \ge \eta\Big\}. \tag{79}$$

---

[5] Strictly speaking, this assumption is not required for our further development. On the other hand, it seems to us that nothing could be gained, had we assumed instead that the default-free bonds market is not complete.

[6] More precisely, this would be the no-arbitrage condition between defaultable bond and savings account, had the process $\tilde{D}^0(t,T)$ represented the price (as opposed to the pre-default value) of a defaultable bond.

[7] This is obvious, if we assume, in addition, that $\sigma(t,T) = \tilde{\sigma}(t,T)$, since then $\lambda_t = g(t,t) - f(t,t)$. A general case eludes us at present. Schönbucher [55]–[56] derives the equality $\phi_t\lambda_t = g(t,t) - f(t,t)$ for a strictly positive process ϕ, but he works in a somewhat different setup.

As one might expect, we extend W^* (and all other previously introduced processes) to the enlarged space by setting $W^*_t(\omega,\hat\omega) = W^*_t(\omega)$, etc. Then, the desired properties are easily seen to hold under $\mathbf{Q}^* = \mathbf{P}^* \otimes \mathbf{Q}$. It is worthwhile to notice that for obvious reasons we cannot require the stopping time τ to be independent of W^* (unless the process λ is constant).

We are in a position to introduce the price process of a T-maturity defaultable bond. Let us first define an auxiliary process $\hat Z(t,T)$ by setting

$$d\hat Z(t,T) = \hat Z(t,T)\tilde b(t,T)\cdot dW^*_t - \hat Z(t-,T)\,dM_t. \tag{80}$$

For obvious reasons, $\hat Z(t,T)$ follows a local martingale under $\mathbf{Q}^*$. Moreover, we may rewrite (80) as follows

$$d\hat Z(t,T) = \hat Z(t,T)\big(\lambda_t\,dt + \tilde b(t,T)\cdot dW^*_t\big) - \hat Z(t-,T)\,dH_t. \tag{81}$$

The last equality follows from (78), combined with the fact that $\hat Z(t,T)$ vanishes on the random set $[\tau,T]$ (and thus $\lambda_t\,\mathrm{I}_{\{t\le\tau\}}$ may be replaced with λ_t). A comparison of (75) and (81) shows that

$$\hat Z(t,T) = \mathrm{I}_{\{\tau>t\}}\tilde Z(t,T). \tag{82}$$

We define the price process of a T-maturity defaultable bond by setting

$$D^0(t,T) := B_t\hat Z(t,T) = \mathrm{I}_{\{\tau>t\}}\tilde D^0(t,T), \tag{83}$$

where the second equality is a consequence of (82).

Remark 6.1 The necessity of enlarging the underlying is closely related to the fact that it is not possible to replicate a defaultable bond using risk-free bonds. Put another way, the process $\tilde D^0(t,T)$ does not correspond to the wealth of a self-financing portfolio of risk-free bonds (i.e., it does not represent a redundant security in the risk-free bonds market). On the other hand, a defaultable bond $D^0(t,T)$ is redundant on the random set $[0,\tau[$, i.e., before the default time (which is a rather weak statement, however, since the stopping time τ is not accessible).

Proposition 6.1 *Let* $D^0(t,T)$ *be given by formula* (83). *Then the dynamics of* $D^0(t,T)$ *are*

$$D^0(t,T) = D^0(t,T)\Big(\big(\tilde a(t,T) + \tilde b(t,T)\gamma_t\big)\,dt + \tilde b(t,T)\cdot dW^*_t\Big) - D^0(t-,T)\,dH_t,$$

and

$$D^0(t,T) = D^0(t,T)\Big(\tilde a(t,T)\,dt + \tilde b(t,T)\cdot dW^*_t\Big) - D^0(t-,T)\,dH_t,$$

under $\mathbf{Q}^*$ *and under* $\mathbf{P}$, *respectively. The risk-neutral valuation formula holds under* $\mathbf{Q}^*$, *that is,*

$$D^0(t,T) = B_t\,\mathbf{E}_{\mathbf{Q}^*}(B_T^{-1}\,\mathrm{I}_{\{T<\tau\}}\,|\,\mathcal{F}_t). \tag{84}$$

Proof. The first statement is an immediate consequence of definition (83), combined with (75) and (81)–(82). From (76), we get

$$d\tilde D^0(t,T) = \tilde D^0(t,T)\big((r_t+\lambda_t)\,dt + \tilde b(t,T)\cdot dW^*_t\big), \tag{85}$$

so that (recall that $\tilde D^0(T,T) = 1$)

$$\tilde D^0(t,T) = \tilde B_t\,\mathbf{E}_{\mathbf{Q}^*}(\tilde B_T^{-1}\,|\,\mathcal{F}_t)$$

with (cf. (20))

$$\tilde B_t = \exp\Big(\int_0^t (r_u+\lambda_u)\,du\Big).$$

This means that $\tilde D^0(t,T)$ corresponds to the process V introduced in Proposition 2.1 (with $Z=0$ and $X=1$). Since $\Delta V_\tau = 0$ (this holds since we know that the process $\tilde D^0(t,T)$ is continuous), using Corollary 2.1, we obtain

$$\mathrm{I}_{\{\tau>t\}}\tilde D^0(t,T) = B_t\,\mathbf{E}_{\mathbf{Q}^*}(B_T^{-1}\,\mathrm{I}_{\{T<\tau\}}\,|\,\mathcal{F}_t).$$

In view of (83), this proves (84). $\square$

References

[1] Artzner, P. and Delbaen, F. (1995) “Default risk insurance and incomplete markets.” *Mathematical Finance* 5, 187–195.

[2] Arvanitis, A., Gregory, J. and Laurent, J.-P. (1998) “Building models for credit spreads.” Working paper, Paribas and CREST.

[3] Black, F. and Cox, J.C. (1976) “Valuing corporate securities: some effects of bond indenture provisions.” *Journal of Finance* 31, 351–367.

[4] Brémaud, P. (1980) *Point Processes and Queues - Martingale Dynamics.* Springer, Heidelberg Berlin New York.

[5] Brennan, M.J. and Schwartz, E.S. (1977) “Convertible bonds: valuation and optimal strategies for call and conversion.” *Journal of Finance* 32, 1699–1715.

[6] Brennan, M.J. and Schwartz, E.S. (1978) “Corporate income taxes, valuation and the problem of optimal capital structure.” *Journal of Business* 51, 103–114.

[7] Brennan, M.J. and Schwartz, E.S. (1980) “Analyzing convertible bonds.” *Journal of Financial and Quantitative Analysis* 15, 907–929.

[8] Briys, E. and de Varenne, F. (1997) “Valuing risky fixed rate debt: an extension.” *Journal of Financial and Quantitative Analysis* 32, 239–248.

[9] Cooper, I. and Martin, M. (1996) “Default risk and derivative products.” *Applied Mathematical Finance* 3, 53–74.

[10] Cooper, I. and Mello, A. (1991) “The default risk of swaps.” *Journal of Finance* 46, 597–620.

[11] Cooper, I. and Mello, A. (1992) “Pricing and optimal use of forward contracts with default risk.” Working paper, London School of Economics.

[12] Crouhy, M., Galai, D. and Mark, R. (1998) “Credit risk revisited.” *Risk - Credit Risk Supplement,* March, 40–44.

[13] Das, S.R. and Tufano, P. (1996) “Pricing credit-sensitive debt when interest rates, credit ratings, and credit spreads are stochastic.” *Journal of Financial Engineering* 5(2), 161–198.

[14] Duffee, G. (1996) “On measuring credit risks of derivative instruments.” *Journal of Banking and Finance* 20, 805–833.

[15] Duffee, G. (1998) “The relation between Treasury yields and corporate bond yield spreads.” Forthcoming in *Journal of Finance.*

[16] Duffie, D. (1994) “Forward rate curves with default risk.” Working paper, Stanford University.

[17] Duffie, D. (1998) “First-to-default valuation.” Working paper, Stanford University.

[18] Duffie, D. (1998) “Defaultable term structure models with fractional recovery of par.” Working paper, Stanford University.

[19] Duffie, D. and Huang, M. (1996) “Swap rates and credit quality.” *Journal of Finance* 51, 921–949.

[20] Duffie, D. and Lando, D. (1998) “The term structure of credit spreads with incomplete accounting data.” Working paper, Stanford University and University of Copenhagen.

[21] Duffie, D., Schroder, M. and Skiadas, C. (1996) “Recursive valuation of defaultable securities and the timing of resolution of uncertainty.” *Annals of Applied Probability* 6, 1075–1090.

[22] Duffie, D. and Singleton, K. (1994) “Modeling term structures of defaultable bonds.” Working paper, Stanford University (current version, July 1998).

[23] Duffie, D. and Singleton, K. (1997) “An econometric model of the term structure of interest rate swap yields.” *Journal of Finance* 52, 1287–1321.

[24] Duffie, D. and Singleton, K. (1998a) “Ratings-based term structures of credit spreads.” Working paper, Stanford University.

[25] Duffie, D. and Singleton, K. (1998b) “Simulating correlated defaults.” Working paper, Stanford University.

[26] Francois, P. (1996) “Bond evaluation with default risk: a review of continuous time approach.” Working paper, ESSEC.

[27] Galai, D. and Masulis, R.W. (1976) “The option pricing model and the risk factor of stock.” *Journal of Financial Economics* 3, 53–81.

[28] Gauthier, C., de la Noue, P. and Rouzeau, E. (1998) “Analyzing corporate credit risk: a quantitative approach.” *Quants No.29*, Recherche and Innovation, Crédit Commercial de France.

[29] Heath, D., Jarrow, R. and Morton, A. (1992) “Bond pricing and the term structure of interest rates: A new methodology for contingent claim valuation.” *Econometrica* 60, 77–105.

[30] Huge, B. and Lando, D. (1998) “Swap pricing with two-sided default risk in a rating-based model.” Working paper, University of Copenhagen.

[31] Hull, J.C. and White, A. (1995) “The impact of default risk on the prices of options and other derivative securities.” *Journal of Banking and Finance* 19, 299–322.

[32] Jarrow, R., Lando, D. and Turnbull, S. (1997) “A Markov model for the term structure of credit risk spreads.” *Review of Financial Studies* 10(2), 481–523.

[33] Jarrow, R. and Madan, D.B. (1995) “Option pricing using the term structure of interest rates to hedge systematic discontinuities in asset returns.” *Mathematical Finance* 5, 311–336.

[34] Jarrow, R. and Turnbull, S.M. (1995) “Pricing derivatives on financial securities subject to credit risk.” *Journal of Finance* 50, 53–85.

[35] Johnson, H. and Stulz, R. (1987) “The pricing of options with default risk.” *Journal of Finance* 42, 267–280.

[36] Johnson, R.E. (1967) “Term structure of corporate bond yields as a function of risk of default.” *Journal of Finance* 22, 313–345.

[37] Jonkhart, M.J.L (1979) “On the term structure of interest rates and the risk of default: an analytical approach.” *Journal of Banking and Finance* 3, 253–262.

[38] Kijima, M. (1998) “Monotonicity in a Markov chain model for valuing coupon bond subject to credit risk.” *Mathematical Finance* 8, 229–247.

[39] Kim, I.J., Ramaswamy, K. and Sundaresan, S. (1993) “Does default risk in coupons affect the valuation of corporate bonds?” *Financial Management* 22, 117–131.

[40] Lando, D. (1997) “Modelling bonds and derivatives with credit risk.” *Mathematics of Derivative Securities,* M.Dempster, S.Pliska, eds. Cambridge University Press, Cambridge, pp. 369–393.

[41] Lando, D. (1998), "On Cox processes and credit-risky securities." Working paper, University of Copenhagen.

[42] Leland, H.E. (1994) "Corporate debt value, bond covenants, and optimal capital structure." *Journal of Finance* 49, 1213–1252.

[43] Leland, H.E. and Toft, K. (1996) "Optimal capital structure, endogenous bankruptcy, and the term structure of credit spreads." *Journal of Finance* 51, 987–1019.

[44] Litterman, R. and Iben, T. (1988) "Corporate bond valuation and the term structure of credit spreads." Working paper, Goldman Sachs.

[45] Longstaff, F.A. and Schwartz, E.S. (1995) "A simple approach to valuing risky fixed and floating rate debt." *Journal of Finance* 50, 789–819.

[46] Madan, D.B. and Unal, H. (1995) "Pricing the risk of default." Forthcoming in *Reviews of Derivatives Research.*

[47] Mella-Barral, P. and Tychon, P. (1996) "Default risk in asset pricing." Working paper, London School of Economics and Université Catholique de Louvain.

[48] Merton, R.C. (1974) "On the pricing of corporate debt: the risk structure of interest rates." *Journal of Finance* 29, 449–470.

[49] Nielsen, T.N., Saà-Requejo, J. and Santa-Clara, P. (1993) "Default risk and interest rate risk: the term structure of default spreads." Working paper, INSEAD.

[50] Pitts, C. and Selby, M. (1983) "The pricing of corporate debt: a further note." *Journal of Finance* 38, 1311–1313.

[51] Pye, G. (1974) "Gauging the default premium." *Financial Analyst Journal*, 30(1), 49–52.

[52] Ramaswamy, K. and Sundaresan, S.M. (1986) "The valuation of floating-rate instruments, theory and evidence." *Journal of Financial Economics* 17, 251–272.

[53] Rendleman, R.J. (1992) "How risks are shared in interest rate swaps." *Journal of Financial Services Research* 5–34.

[54] Rich, D. (1994) "A note on the valuation and hedging of equity swaps." *Journal of Financial Engineering* 5, 323–334.

[55] Schönbucher, P.J. (1996) "The term structure of defaultable bond prices." Working paper, University of Bonn.

[56] Schönbucher, P.J. (1997) "Term structure modelling of defaultable bonds." Working paper, London School of Economics.

[57] Schlögl, L. (1998) "An exposition of intensity-based models of securities and derivatives with default risk." Working paper, University of Bonn.

[58] Shimko, D., Tejima, N. and Van Deventer, D. (1993) "The pricing of risky debt when interest rates are stochastic." *Journal of Fixed Income* 3, 58–65.

[59] Skinner, F.S. (1998) "Hedging bonds subject to credit risk." *Journal of Banking and Finance* 22, 321–345.

[60] Thomas, L.C., Allen, D.E. and Morkel-Kingsbury, N. (1998) "A hidden Markov chain model for the term structure of bond credit risk spreads." Working paper, Edith Cowan University.

[61] Yawitz, J. (1977) "An analytical model of interest rate differentials and different default recoveries." *Journal of Financial and Quantitative Analysis* 12, 481–490.