SYMPECTIC $T_7$, $T_8$ SINGULARITIES AND LAGRANGIAN TANGENCY ORDERS

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Abstract We study the local symplectic algebra of curves. We use the method of algebraic restrictions to classify symplectic $T_7$, $T_8$ singularities. We define discrete symplectic invariants (the Lagrangian tangency orders) and compare them with the index of isotropy. We use these invariants to distinguish symplectic singularities of classical $T_7$ singularity. We also give the geometric description of symplectic classes of the singularity.

Keywords: symplectic manifold; curves; local symplectic algebra; algebraic restrictions; relative Darboux theorem; singularities

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1. Introduction

In this paper we study the symplectic classification of singular curves under the following equivalence.

Definition 1.1. Let $N_1$ and $N_2$ be germs of subsets of symplectic space $(\mathbb{R}^{2n}, \omega)$. $N_1$ and $N_2$ are symplectically equivalent if there exists a symplectomorphism germ

$$\Phi: (\mathbb{R}^{2n}, \omega) \to (\mathbb{R}^{2n}, \omega)$$

such that $\Phi(N_1) = N_2$.

We recall that $\omega$ is a symplectic form if $\omega$ is a smooth non-degenerate closed 2-form, and $\Phi: \mathbb{R}^{2n} \to \mathbb{R}^{2n}$ is a symplectomorphism if $\Phi$ is diffeomorphism and $\Phi^*\omega = \omega$.

Symplectic classification of curves was first studied by Arnold. In [2] he discovered new symplectic invariants of singular curves. He proved that the $A_{2k}$ singularity of a planar curve (the orbit with respect to standard $A$-equivalence of parametrized curves) split into exactly $2k + 1$ symplectic singularities (orbits with respect to symplectic equivalence of parametrized curves). He distinguished different symplectic singularities by different orders of tangency of the parametrized curve to the nearest smooth Lagrangian submanifold. He posed the problem of expressing these invariants in terms of the local algebra’s
interaction with the symplectic structure and he proposed calling this interaction the ‘local symplectic algebra’.

In [12, 13] Ishikawa and Janeczko classified symplectic singularities of curves in the two-dimensional symplectic space. All simple curves in this classification are quasi-homogeneous.

We recall that a subset \( N \) of \( \mathbb{R}^m \) is **quasi-homogeneous** if there exist a coordinate system \((x_1, \ldots, x_m)\) on \( \mathbb{R}^m \) and positive numbers \( w_1, \ldots, w_m \) (called weights) such that, for any point \((y_1, \ldots, y_m)\) \( \in \mathbb{R}^m \) and any \( t \in \mathbb{R} \), if \((y_1, \ldots, y_m)\) belongs to \( N \), then a point \((t^{w_1}y_1, \ldots, t^{w_m}y_m)\) belongs to \( N \).

A symplectic form on a two-dimensional manifold is a special case of a volume form on a smooth manifold. The generalization of results in [12] to volume-preserving classification of singular varieties and maps in arbitrary dimensions was obtained in [9]. The orbit of action of all diffeomorphism germs agrees with volume-preserving orbit or splits into two volume-preserving orbits (in the case \( \mathbb{K} = \mathbb{R} \)) for germs which satisfy a special weak form of quasi-homogeneity, e.g. the weak quasi-homogeneity of varieties is a quasi-homogeneity with non-negative weights \( w_i \geq 0 \) and \( \sum_i w_i > 0 \).

Symplectic singularity is stably simple if it is simple, and remains simple if the ambient symplectic space is symplectically embedded (i.e. as a symplectic submanifold) into a larger symplectic space. In [14] Kolgushkin classified the stably simple symplectic singularities of parametrized curves (in the \( \mathbb{C} \)-analytic category). All stably simple symplectic singularities of curves are also quasi-homogeneous.

In [8] new symplectic invariants of singular quasi-homogeneous subsets of a symplectic space were explained by the algebraic restrictions of the symplectic form to these subsets.

The algebraic restriction is an equivalence class of the following relation on the space of differential \( k \)-forms.

Differential \( k \)-forms \( \omega_1 \) and \( \omega_2 \) have the same **algebraic restriction** to a subset \( N \) if \( \omega_1 - \omega_2 = \alpha + d\beta \), where \( \alpha \) is a \( k \)-form vanishing on \( N \) and \( \beta \) is a \((k-1)\)-form vanishing on \( N \).

The generalization of the Darboux–Givental Theorem [3] to germs of arbitrary subsets of the symplectic space was obtained in [8] (see also [17]). This result reduces the problem of symplectic classification of germs of quasi-homogeneous subsets to the problem of classification of algebraic restrictions of symplectic forms to these subsets. For non-quasi-homogeneous subsets there is one more cohomological invariant apart from the algebraic restriction [7, 8]. The dimension of the space of algebraic restrictions of closed 2-forms to a one-dimensional quasi-homogeneous isolated complete intersection singularity \( C \) is equal to the multiplicity of \( C \) [8]. In [6] it was proved that the space of algebraic restrictions of closed 2-forms to a one-dimensional (singular) analytic variety is finite dimensional. In [8] the method of algebraic restrictions was applied to various classification problems in a symplectic space. In particular, the complete symplectic classification of classical \( A–D–E \) singularities of planar curves and the \( S_5 \) singularity were obtained. Most of the different symplectic singularity classes were distinguished by new discrete symplectic invariants: the index of isotropy and the symplectic multiplicity.
In this paper, following ideas from \cite{2,6}, we use new discrete symplectic invariants: the Lagrangian tangency orders (see §2.1). Although this invariant has a similar definition to the index of isotropy, its nature is different. Since the Lagrangian tangency order takes into account the weights of quasi-homogeneity of curves, it allows us to distinguish more symplectic classes in many cases. For example, using the Lagrangian tangency order, we are able to distinguish classes $E_6^3$ and $E_6^{4,\pm}$ of classical planar singularity $E_6$, which cannot be distinguished by the isotropy index or by the symplectic multiplicity. We also present other examples of singularities which can be distinguished only by the Lagrangian tangency order. On the other hand, there are singularities for which symplectic classes can be distinguished by the index of isotropy but not by the Lagrangian tangency order, for example, the parametric curve with semigroup $(3, 7, 11)$ and $T_8$ singularity. These examples show that there are no simple relations between the Lagrangian tangency order and the index of isotropy, even for the case of parametric curves.

We also obtain the complete symplectic classification of the classical isolated complete intersection singularity $T_7$ using the method of algebraic restrictions (Theorem 3.1). We calculate discrete symplectic invariants for this classification (Theorems 3.3) and we present geometric descriptions of its symplectic orbits (Theorem 3.5).

The paper is organized as follows. In §2 we present known discrete symplectic invariants and introduce the Lagrangian tangency orders. We also compare the Lagrangian tangency order and the index of isotropy. Symplectic classification of $T_7$ singularity is studied in §3. In §4 we recall the method of algebraic restrictions and use it to classify $T_7$ symplectic singularities.

2. Discrete symplectic invariants

We define discrete symplectic invariants to distinguish symplectic singularity classes. The first one is the symplectic multiplicity \cite{8} introduced in \cite{12} as a symplectic defect of a curve.

Let $N$ be a germ of a subset of $(\mathbb{R}^{2n}, \omega)$.

**Definition 2.1.** The *symplectic multiplicity* $\mu_{\text{sympl}}(N)$ of $N$ is the codimension of a symplectic orbit of $N$ in an orbit of $N$ with respect to the action of the group of local diffeomorphisms.

The second one is the index of isotropy \cite{8}.

**Definition 2.2.** The *index of isotropy* $\text{ind}(N)$ of $N$ is the maximal order of vanishing of the 2-forms $\omega|_{TM}$ over all smooth submanifolds $M$ containing $N$.

This invariant has geometrical interpretation. An equivalent definition is as follows: the index of isotropy of $N$ is the maximal order of tangency between non-singular submanifolds containing $N$ and non-singular isotropic submanifolds of the same dimension. The index of isotropy is equal to 0 if $N$ is not contained in any non-singular submanifold which is tangent to some isotropic submanifold of the same dimension. If $N$ is contained in a non-singular Lagrangian submanifold, then the index of isotropy is $\infty$. 
Remark 2.3. If $N$ consists of invariant components $C_i$ we can calculate the index of isotropy for each component $\text{ind}(C_i)$ as the maximal order of vanishing of the 2-forms $\omega|_{TM}$ over all smooth submanifolds $M$ containing $C_i$.

The symplectic multiplicity and the index of isotropy can be described in terms of algebraic restrictions (Propositions 4.6 and 4.7).

2.1. Lagrangian tangency order

There is one more discrete symplectic invariant, introduced in [6] (following ideas from [2]), which is defined specifically for a parametrized curve. This is the maximal tangency order of a curve $f: \mathbb{R} \to M$ to a smooth Lagrangian submanifold. If $H_1 = \cdots = H_n = 0$ define a smooth submanifold $L$ in the symplectic space, then the tangency order of a curve $f: \mathbb{R} \to M$ to $L$ is the minimum of the orders of vanishing at 0 of functions $H_1 \circ f, \ldots, H_n \circ f$. We denote the tangency order of $f$ to $L$ by $t(f, L)$.

Definition 2.4. The Lagrangian tangency order $Lt(f)$ of a curve $f$ is the maximum of $t(f, L)$ over all smooth Lagrangian submanifolds $L$ of the symplectic space.

The Lagrangian tangency order of a quasi-homogeneous curve in a symplectic space can also be expressed in terms of algebraic restrictions (Proposition 4.8).

We can generalize this invariant for curves which may be parametrized analytically. Lagrangian tangency order is the same for every ‘good’ analytic parametrization of a curve [16]. Considering only such parametrizations, we can choose one and calculate the invariant for it. It is easy to show that this invariant does not depend on chosen parametrization.

Proposition 2.5. Let $f: \mathbb{R} \to M$ and $g: \mathbb{R} \to M$ be good analytic parametrizations of the same curve. Then $Lt(f) = Lt(g)$.

Proof. There exists a diffeomorphism $\theta: \mathbb{R} \to \mathbb{R}$ such that $g(s) = f(\theta(s))$ and $\frac{d\theta}{ds}|_0 \neq 0$. Let $H_1 = \cdots = H_n = 0$ define a smooth submanifold $L$ in the symplectic space. If $d^i(H_i \circ f)/dt^i|_0 = 0$ for $i = 1, \ldots, k$, then

$$
\frac{d^{k+1}(H_i \circ g)}{ds^{k+1}}|_0 = \frac{d^{k+1}(H_i \circ f \circ \theta)}{ds^{k+1}}|_0 = \left. \frac{d^{k+1}(H_i \circ f)}{dt^{k+1}} \right|_0 \cdot \left( \frac{d\theta}{ds} \right)^{k+1}|_0,
$$

so the orders of vanishing at 0 of functions $H_i \circ f$ and $H_i \circ g$ are equal, and hence $t(f, L) = t(g, L)$, which implies that $Lt(f) = Lt(g)$. \qed

We can generalize Lagrangian tangency order for sets containing parametric curves. Let $N$ be a subset of a symplectic space $(\mathbb{R}^{2n}, \omega)$.

Definition 2.6. The tangency order of the germ of a subset $N$ to the germ of a submanifold $L$ $t[N, L]$ is equal to the minimum of $t(f, L)$ over all parametrized curve-germs $f$ such that $\text{Im} f \subseteq N$.

Definition 2.7. The Lagrangian tangency order of $N$, $Lt(N)$, is equal to the maximum of $t[N, L]$ over all smooth Lagrangian submanifold-germs $L$ of the symplectic space.
In this paper we consider $N$ which are singular analytic curves. They may be identified with a multi-germ of parametric curves. We define invariants which are special cases of the above definition.

Consider a multi-germ $(f_i)_{i \in \{1, \ldots, r\}}$ of analytically parametrized curves $f_i$. For any smooth submanifold $L$ in the symplectic space we have $r$-tuples $(t(f_1, L), \ldots, t(f_r, L))$.

**Definition 2.8.** For any $I \subseteq \{1, \ldots, r\}$ we define the tangency order of the multi-germ $(f_i)_{i \in I}$ to $L$:

$t((f_i)_{i \in I}, L) = \min_{i \in I} t(f_i, L)$.

**Definition 2.9.** The Lagrangian tangency order $L^t((f_i)_{i \in I})$ of a multi-germ $(f_i)_{i \in I}$ is the maximum of $t((f_i)_{i \in I}, L)$ over all smooth Lagrangian submanifolds $L$ of the symplectic space.

For multi-germs we can also define relative invariants according to selected branches or collections of branches.

**Definition 2.10.** Let $S \subseteq I \subseteq \{1, \ldots, r\}$. For $i \in S$ let us fix numbers $t_i \leq L^t(f_i)$. The relative Lagrangian tangency order $L^t((f_i)_{i \in I} : (S, (t_i)_{i \in S}))$ of a multi-germ $(f_i)_{i \in I}$ related to $S$ and $(t_i)_{i \in S}$ is the maximum of $t((f_i)_{i \in I \setminus S}, L)$ over all smooth Lagrangian submanifolds $L$ of the symplectic space for which $t(f_i, L) = t_i$, if such submanifolds exist, or $-\infty$ if there are no such submanifolds.

We can also define special relative invariants according to selected branches of the multi-germ.

**Definition 2.11.** For fixed $j \in I$ the Lagrangian tangency order related to $f_j$ of a multi-germ $(f_i)_{i \in I}$ denoted by $L^t((f_i)_{i \in I} : f_j)$ is the maximum of $t((f_i)_{i \in I \setminus \{j\}}, L)$ over all smooth Lagrangian submanifolds $L$ of the symplectic space for which $t(f_j, L) = L^t(f_j)$.

These invariants have geometric interpretations. If $L^t(f_i) = \infty$, then a branch $f_i$ is included in a smooth Lagrangian submanifold. If $L^t((f_i)_{i \in I}) = \infty$, then there exists a Lagrangian submanifold containing all the curves $f_i$ for $i \in I$.

We may use these invariants to distinguish symplectic singularities.

**2.2. Comparison of the Lagrangian tangency order and the index of isotropy**

Definitions of the Lagrangian tangency order and the index of isotropy are similar. They show how far a variety $N$ is from the nearest non-singular Lagrangian submanifold.
Table 2. Symplectic invariants of $D_k$ singularity.

(The branch $C_1$ has a form $(t, 0, 0, 0, \ldots, 0)$. If $k$ is odd, then $C_2$ has a form $(t^{k-2}, f(t), t^2, 0, \ldots, 0)$ and $\lambda_k = 1$. If $k$ is even, then $C_2$ consists of two branches: $B_{\pm} : (\pm t^{(k-2)/2}, f(t), t, 0, \ldots, 0)$ and $\lambda_k = \frac{1}{2}$.)

<table>
<thead>
<tr>
<th>normal form</th>
<th>$f(t)$</th>
<th>$Lt(N)$</th>
<th>$Lt(C_2)$</th>
<th>ind</th>
<th>ind_2</th>
</tr>
</thead>
<tbody>
<tr>
<td>$D_k^0$</td>
<td>$t^{2\lambda_k}$</td>
<td>$2\lambda_k$</td>
<td>$(k-2)\lambda_k$</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>$D_k^1$</td>
<td>$bt^{k\lambda_k} + \frac{1}{4}t^{4\lambda_k}$</td>
<td>$k\lambda_k$</td>
<td>$k\lambda_k$</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>$D_k^i$ ($1 &lt; i &lt; k - 3$)</td>
<td>$bt^{k\lambda_k} + \frac{1}{i+1}t^{2(i+1)\lambda_k}$, $b \neq 0$</td>
<td>$k\lambda_k$</td>
<td>$(k-2+2i)\lambda_k$</td>
<td>1</td>
<td>$i$</td>
</tr>
<tr>
<td>$D_k^{k-3, \pm}$</td>
<td>$(\pm 1)^k t^{k\lambda_k} + \frac{b}{k-2} t^{2(k-2)\lambda_k}$</td>
<td>$k\lambda_k$</td>
<td>$\infty$</td>
<td>1</td>
<td>$\infty$</td>
</tr>
<tr>
<td>$D_k^{k-2}$</td>
<td>$\frac{1}{k-2} t^{2(k-2)\lambda_k}$</td>
<td>$(3k-8)\lambda_k$</td>
<td>$\infty$</td>
<td>$k-3$</td>
<td>$\infty$</td>
</tr>
<tr>
<td>$D_k^{k-1}$</td>
<td>$\frac{1}{k-1} t^{2(k-1)\lambda_k}$</td>
<td>$(3k-6)\lambda_k$</td>
<td>$\infty$</td>
<td>$k-2$</td>
<td>$\infty$</td>
</tr>
<tr>
<td>$D_k^k$</td>
<td>0</td>
<td>$\infty$</td>
<td>$\infty$</td>
<td>$\infty$</td>
<td>$\infty$</td>
</tr>
</tbody>
</table>

The index of isotropy of a quasi-homogeneous set $N$ is $\infty$ if and only if the Lagrangian tangency order of $N$ is $\infty$. Studying classical singularities, we have found examples of all possible interactions between these invariants.

Example 2.12. For some singularities the index of isotropy distinguishes the same symplectic classes that can be distinguished by the Lagrangian tangency order. It is observed, for example, for planar curves: the classical $A_k$ and $D_k$ singularities (Tables 1 and 2) and for $S_\mu$ singularities studied in [10].

A complete symplectic classification of classical $A-D-E$ singularities of planar curves was obtained using a method of algebraic restriction in [8]. Below, we compare the Lagrangian tangency order and the index of isotropy for these singularities. A curve $N$ may be described as a parametrized curve or as a union of parametrized components $C_i$ preserved by local diffeomorphisms in the symplectic space $(\mathbb{R}^{2n}, \omega_0 = \sum_{i=1}^{n} dp_i \wedge dq_i)$, $n \geq 2$. For calculating the Lagrangian tangency orders, we give their parametrization in the coordinate system $(p_1, q_1, p_2, q_2, \ldots, p_n, q_n)$.

Denote by $(A_k)$ the class of varieties in a fixed symplectic space $(\mathbb{R}^{2n}, \omega)$ which are diffeomorphic to

$$A_k = \{x \in \mathbb{R}^{2n} : x_1^{k+1} - x_2^2 - x_{\geq 3} = 0\}. \quad (2.1)$$

A curve $N \in (A_k)$ can be described as a parametrized singular curve $C$ for $k$ even, or as a pair of two smooth parametrized branches $B_+$ and $B_-$ if $k$ is odd. We denote $Lt(C)$ or $Lt(B_+, B_-)$, respectively, by $Lt(N)$.
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Table 3. Symplectic invariants of $E_6$ singularity.

<table>
<thead>
<tr>
<th>form</th>
<th>parametrization</th>
<th>$Lt(N)$</th>
<th>ind</th>
<th>$\mu^{\text{symp}}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$E_6^0$</td>
<td>$(t^4, t^3, t^3, 0, \ldots, 0)$</td>
<td>4</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>$E_6^1$</td>
<td>$(t^4, \pm \frac{1}{2}t^6 + bt^7, t^3, 0, \ldots, 0)$</td>
<td>7</td>
<td>1</td>
<td>2</td>
</tr>
<tr>
<td>$E_6^2$</td>
<td>$(t^4, t^7 + \frac{1}{4}bt^9, t^3, 0, \ldots, 0)$</td>
<td>8</td>
<td>1</td>
<td>3</td>
</tr>
<tr>
<td>$E_6^3$</td>
<td>$(t^4, \frac{1}{1}t^9 + \frac{1}{4}bt^{10}, t^3, 0, \ldots, 0)$</td>
<td>10</td>
<td>2</td>
<td>4</td>
</tr>
<tr>
<td>$E_6^4$</td>
<td>$(t^4, \pm \frac{1}{4}t^{10}, t^3, 0, \ldots, 0)$</td>
<td>11</td>
<td>2</td>
<td>4</td>
</tr>
<tr>
<td>$E_6^5$</td>
<td>$(t^4, \frac{1}{4}t^{13}, t^3, 0, \ldots, 0)$</td>
<td>14</td>
<td>3</td>
<td>5</td>
</tr>
<tr>
<td>$E_6^6$</td>
<td>$(t^4, 0, t^3, 0, \ldots, 0)$</td>
<td>$\infty$</td>
<td>$\infty$</td>
<td>6</td>
</tr>
</tbody>
</table>

Denote by $(D_k)$ for $k \geq 4$ the class of varieties in a fixed symplectic space $(\mathbb{R}^{2n}, \omega)$ which are diffeomorphic to

$$D_k = \{x \in \mathbb{R}^{2n+4}: x_1^2x_2 - x_2^{k-1} = x_{\geq 3} = 0\}. \quad (2.2)$$

A curve $N \in (D_k)$ consists of two invariant components: $C_1$ (smooth) and $C_2$ (singular diffeomorphic to $A_{k-3}$). $C_2$ may consist of one or two branches, depending on $k$. To distinguish the symplectic classes completely we need two invariants: $Lt(N)$ (the Lagrangian tangency order of $N$) and $Lt(C_2)$ (the Lagrangian tangency order of the singular component $C_2$). Equivalently, we can use the index of isotropy of $N$, ind, and the index of isotropy of $C_2$, ind$_2$.

**Example 2.13.** There are also symplectic singularities distinguished by the Lagrangian tangency order but not by the index of isotropy. The simplest example is planar singularity $E_6$ (Table 3). We also observe such a ‘greater sensitivity’ of the Lagrangian tangency order for $E_7$ and $E_8$ singularities and for parametric curves with the semigroups $(3, 4, 5)$, $(3, 5, 7)$ and $(3, 7, 8)$ studied in [6].

Denote by $(E_6)$ the class of varieties in a fixed symplectic space $(\mathbb{R}^{2n}, \omega)$ which are diffeomorphic to

$$E_6 = \{x \in \mathbb{R}^{2n+4}: x_1^3 - x_2^4 = x_{\geq 3} = 0\}. \quad (2.3)$$

As can be seen in Table 3, we are able, by the Lagrangian tangency order, to distinguish the classes $E_6^3$ and $E_6^{4, \pm}$ which cannot be distinguished by the index of isotropy or by the symplectic multiplicity.

**Example 2.14.** Some symplectic singularities can be distinguished by the index of isotropy but not by the Lagrangian tangency order. We observe such a situation for a parametric quasi-homogeneous curve-germ with semigroup $(3, 7, 11)$ listed as a stably simple singularity of curves in [1]. Another example is the $T_8$ singularity presented below (see the rows for $(T_8)^4$ and $(T_8)^{6,2}$ in Table 6).

The germ of a curve $f: (\mathbb{R}, 0) \rightarrow (\mathbb{R}^{2n}, 0)$ with semigroup $(3, 7, 11)$ is diffeomorphic to the curve $t \rightarrow (t^3, t^7, t^{11}, 0, \ldots, 0)$. Among symplectic singularities of this curve-germ
in the symplectic space \((\mathbb{R}^{2n}, \omega = \sum_{i=1}^{n} dp_i \wedge dq_i)\) with the canonical coordinates \((p_1, q_1, \ldots, p_n, q_n)\) we have, for example, the classes represented by the normal forms given in Table 4.

Symplectic classes (1) and (3) have the same Lagrangian tangency order (equal to 10) but have different indices of isotropy (1 and 0, respectively). Symplectic classes (2) and (3) have the same index of isotropy (equal to 0) but have different Lagrangian tangency orders (11 and 10, respectively). We also observe that the Lagrangian tangency order for class (1) is less than that for class (2) but the inverse inequality is satisfied for the indices of isotropy.

Another example is \(T_8\) singularity. Denote by \((T_8)\) the class of varieties in a fixed symplectic space \((\mathbb{R}^{2n}, \omega)\) which are diffeomorphic to

\[
T_8 = \{ x \in \mathbb{R}^{2n+4}: x^2 + x^3 - x^4 = x_2x_3 = x_{\geq 4} = 0 \}. \tag{2.4}
\]

This is the classical one-dimensional isolated complete intersection singularity \(T_8\) [5, 11].

Let \(N \in (T_8)\). \(N\) is quasi-homogeneous with weights \(w(x_1) = 6, w(x_2) = 4, w(x_3) = 3\). A curve \(N\) consists of two invariant singular components: \(C_1\) (diffeomorphic to the \(A_2\) singularity) and \(C_2\) (diffeomorphic to the \(A_3\) singularity), which is a union of two smooth branches \(B_+\) and \(B_-\). In local coordinates they have the form

\[
C_1 = \{ x_1^2 + x_2^3 = 0, x_3 = x_{\geq 4} = 0 \}, \quad B_\pm = \{ x_1 \pm x_2^3 = 0, x_2 = x_{\geq 4} = 0 \}.
\]

Using the method of algebraic restrictions, one can obtain, in the same way as presented in the last two sections for the case of the \(T_7\) singularity, the following complete classification of symplectic \(T_8\) singularities.

**Theorem 2.15.** Any stratified submanifold of the symplectic space \((\mathbb{R}^{2n}, \omega = \sum_{i=1}^{n} dp_i \wedge dq_i)\) which is diffeomorphic to \(T_8\) is symplectically equivalent to one and only one of the normal forms \((T_8)^i\), \(i = 0, 1, \ldots, 8\). The parameters \(c, c_1, c_2, c_3\) of the normal forms are moduli:

\[
\begin{align*}
T_8^0: \quad & p_1^2 + p_2^3 - q_1^4 = 0, \quad p_2q_1 = 0, \quad q_2 = c_1q_1 - c_2p_1p_{\geq 3} = q_{\geq 3} = 0, \\
& c_1 \cdot c_2 \neq 0; \\
T_8^1: \quad & p_1^2 + p_2^3 - q_1^4 = 0, \quad p_2q_1 = 0, \quad q_2 = c_1q_1 - c_2p_1 - c_3p_1p_2, \quad p_{\geq 3} = q_{\geq 3} = 0, \\
& c_1 \cdot c_2 = 0;
\end{align*}
\]
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\[ T_{8}^{13} : p_1^2 + q_1^3 - q_4^4 = 0, \quad q_1q_2 = 0, \quad p_2 = c_1q_1 + c_2p_1q_2, \quad p_{\geq 3} = q_{\geq 3} = 0, \]

\[ c_1 \cdot c_2 \neq 0; \]

\[ T_{8}^{23} : p_1^2 + q_1^3 - q_4^4 = 0, \quad q_1q_2 = 0, \quad p_2 = c_1q_1 + c_2p_1q_2 + c_3p_1q_4^2, \quad p_{\geq 3} = q_{\geq 3} = 0, \]

\[ c_1 \cdot c_2 = 0; \]

\[ T_{8}^{2>3} : p_1^2 + p_1^3 - q_4^4 = 0, \quad p_1q_1 = 0, \quad q_2 = \frac{1}{2}c_1q_1^2 + \frac{1}{2}c_2p_1^2, \quad p_{\geq 3} = q_{\geq 3} = 0, \]

\[ c_1 \neq 0; \]

\[ T_{8}^{3,0} : p_1^2 + p_1^3 - q_4^4 = 0, \quad p_1q_1 = 0, \quad q_2 = \frac{1}{2}c_1p_1^2 + \frac{1}{3}c_2q_1^3, \quad p_{\geq 3} = q_{\geq 3} = 0, \]

\[ (c_1, c_2) \neq (0, 0); \]

\[ T_{8}^{5,0} : p_1^2 + p_1^3 - q_4^4 = 0, \quad p_1q_1 = 0, \quad q_2 = \frac{1}{2}c_1q_1^2, \quad p_{\geq 3} = q_{\geq 3} = 0; \]

\[ T_{8}^{3,1} : p_1^2 + p_1^3 - q_4^4 = 0, \quad p_2p_3 = 0, \quad q_1 = \frac{1}{2}p_2^2 + \frac{1}{2}c_2p_2^2, \quad q_2 = -c_1p_1p_3, \quad p_{\geq 4} = q_{\geq 3} = 0; \]

\[ T_{8}^{4} : p_1^2 + p_3^3 - q_4^4 = 0, \quad p_2p_3 = 0, \quad q_1 = \frac{1}{2}c_1p_2^2 + \frac{4}{3}c_2p_3^3, \quad q_2 = -p_1p_3, \quad p_{\geq 4} = q_{\geq 3} = 0, \]

\[ (c_1, c_2) \neq (0, 0); \]

\[ T_{8}^{6,1} : p_1^2 + p_3^3 - q_4^4 = 0, \quad p_2p_3 = 0, \quad q_1 = \frac{1}{2}c_2p_3^3, \quad q_2 = -p_1p_3, \quad p_{\geq 4} = q_{\geq 3} = 0; \]

\[ T_{8}^{5,1} : p_1^2 + p_3^3 - q_4^4 = 0, \quad p_2p_3 = 0, \quad q_1 = \frac{1}{2}p_2^3 + \frac{1}{2}c_2p_3^3, \quad p_{\geq 4} = q_{\geq 2} = 0; \]

\[ T_{8}^{6,2} : p_1^2 + p_3^3 - q_4^4 = 0, \quad p_2p_3 = 0, \quad q_1 = \frac{1}{3}p_3^3, \quad p_{\geq 4} = q_{\geq 2} = 0; \]

\[ T_{8}^{7} : p_1^2 + p_3^3 - q_4^4 = 0, \quad p_2p_3 = 0, \quad q_1 = \frac{1}{3}p_3^4, \quad p_{\geq 4} = q_{\geq 2} = 0; \]

\[ T_{8}^{8} : p_1^2 + p_3^3 - q_4^4 = 0, \quad p_2p_3 = 0, \quad q_{\geq 1} = p_{\geq 4} = 0. \]

Lagrangian tangency orders and indices of isotropy were used to obtain a detailed classification of $(T_8)$. A curve $N \in (T_8)$ may be described as a union of three parametrical branches $C_1$, $B_+$, $B_-$. Their parametrization in the coordinate system $(p_1, q_1, p_2, q_2, \ldots, p_n, q_n)$ is presented in the second column of Tables 5 and 6. To distinguish the classes of this singularity, we need the following three invariants:

(i) $L_1(N) = L_1(C_1, B_+, B_-) = \max_L(\min\{t(C_1, L), t(B_+, L), t(B_-, L)\})$;

(ii) $L_1 = L_1(C_1) = \max_L(t(C_1, L))$;

(iii) $L_2 = L_2(C_2) = \max_L(\min\{t(B_+, L), t(B_-, L)\})$;

here $L$ is a smooth Lagrangian submanifold of the symplectic space.

Branches $B_+$ and $B_-$ are diffeomorphic and are not preserved by all symmetries of $T_8$, so we can use neither $L_1(B_+)$ nor $L_1(B_-)$ as invariants. Considering the triples $(L_1, L_2, L_3)$, we obtain a more detailed classification of symplectic singularities of $T_8$ than the classification given in Theorem 2.15. Some subclasses appear to have a natural geometric interpretation.

We also calculate the index of isotropy of $N \in (T_8)$, denoted by ind, and the indices of isotropy of components $C_1$ and $C_2$, denoted by ind$_1$ and ind$_2$, respectively. In Tables 5 and 6 we present a comparison of the invariants.
Table 5. Symplectic invariants for symplectic classes of $T_8$ singularity when $\omega|_W \neq 0$.

(W is the tangent space to a non-singular three-dimensional manifold in $(\mathbb{R}^{2n+3}, \omega)$ containing $N \in (T_8)_8$.)

<table>
<thead>
<tr>
<th>class</th>
<th>parametrization</th>
<th>conditions</th>
<th>$Lt$</th>
<th>$L_1$</th>
<th>$L_2$</th>
<th>ind</th>
<th>ind$_1$</th>
<th>ind$_2$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$(T_8)^0$</td>
<td>$(t^3, 0, -t^2, -c_2 t^3, 0, \ldots)$</td>
<td>$c_1 \cdot c_2 \neq 0$</td>
<td>2</td>
<td>3</td>
<td>2</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td></td>
<td>$(\pm t^2, t, 0, c_1 t \mp c_2 t^2, 0, \ldots)$</td>
<td></td>
<td>2</td>
<td>3</td>
<td>2</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>$(T_8)^3$</td>
<td>$(t^3, 0, -t^2, -c_2 t^3 + c_3 t^3, 0, \ldots)$</td>
<td>$c_1 = 0$, $c_2 \neq 0$</td>
<td>2</td>
<td>3</td>
<td>2</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td></td>
<td>$(\pm t^2, t, 0, c_1 t \mp c_2 t^2, 0, \ldots)$</td>
<td>$c_2 = 0$, $c_3 \neq 0$</td>
<td>2</td>
<td>5</td>
<td>2</td>
<td>0</td>
<td>1</td>
<td>0</td>
</tr>
<tr>
<td></td>
<td></td>
<td>$c_2 = c_3 = 0$</td>
<td>2</td>
<td>$\infty$</td>
<td>2</td>
<td>0</td>
<td>$\infty$</td>
<td>0</td>
</tr>
<tr>
<td>$(T_8)^4$</td>
<td>$(t^3, -t^3, -c_1 t^2, 0, 0, \ldots)$</td>
<td>$c_1 \cdot c_2 \neq 0$</td>
<td>2</td>
<td>3</td>
<td>3</td>
<td>0</td>
<td>0</td>
<td>1</td>
</tr>
<tr>
<td></td>
<td>$(\pm t^2, 0, \pm c_2 t^3, t, 0, \ldots)$</td>
<td></td>
<td>2</td>
<td>3</td>
<td>3</td>
<td>0</td>
<td>0</td>
<td>1</td>
</tr>
<tr>
<td>$(T_8)^5$</td>
<td>$(t^3, -t^2, -c_1 t^2, 0, 0, \ldots)$</td>
<td>$c_1 = 0$, $c_2 \neq 0$</td>
<td>2</td>
<td>3</td>
<td>3</td>
<td>0</td>
<td>0</td>
<td>1</td>
</tr>
<tr>
<td></td>
<td>$(\pm t^2, 0, \pm c_2 t^3, t, 0, \ldots)$</td>
<td>$c_2 = 0$, $c_3 \neq 0$</td>
<td>2</td>
<td>3</td>
<td>3</td>
<td>0</td>
<td>0</td>
<td>1</td>
</tr>
<tr>
<td></td>
<td></td>
<td>$c_2 = c_3 = 0$</td>
<td>2</td>
<td>3</td>
<td>$\infty$</td>
<td>0</td>
<td>0</td>
<td>$\infty$</td>
</tr>
<tr>
<td>$(T_8)^6$</td>
<td>$(-t^2, 0, t^3, 0, 0, \ldots)$</td>
<td>$c_1 \cdot c_2 \neq 0$</td>
<td>2</td>
<td>5</td>
<td>3</td>
<td>0</td>
<td>0</td>
<td>1</td>
</tr>
<tr>
<td></td>
<td>$(0, t, \pm t^2, -\frac{1}{2} c_1 t^4, 0, \ldots)$</td>
<td>$c_1 \neq 0$, $c_2 = 0$</td>
<td>2</td>
<td>5</td>
<td>3</td>
<td>0</td>
<td>0</td>
<td>1</td>
</tr>
<tr>
<td>$(T_8)^7$</td>
<td>$(-t^2, 0, t^3, 0, 0, \ldots)$</td>
<td>$c_1 \cdot c_2 \neq 0$</td>
<td>2</td>
<td>5</td>
<td>4</td>
<td>0</td>
<td>1</td>
<td>2</td>
</tr>
<tr>
<td></td>
<td>$(0, t, \pm t^2, -\frac{1}{2} c_1 t^4, 0, \ldots)$</td>
<td>$c_1 \neq 0$, $c_2 = 0$</td>
<td>2</td>
<td>5</td>
<td>4</td>
<td>0</td>
<td>1</td>
<td>2</td>
</tr>
<tr>
<td></td>
<td></td>
<td>$c_1 = 0$, $c_2 \neq 0$</td>
<td>2</td>
<td>5</td>
<td>$\infty$</td>
<td>0</td>
<td>1</td>
<td>$\infty$</td>
</tr>
<tr>
<td>$(T_8)^8$</td>
<td>$(-t^2, 0, t^3, 0, 0, \ldots)$</td>
<td></td>
<td>2</td>
<td>$\infty$</td>
<td>$\infty$</td>
<td>0</td>
<td>0</td>
<td>$\infty$</td>
</tr>
<tr>
<td></td>
<td>$(0, t, \pm t^2, -\frac{1}{2} c_1 t^4, 0, \ldots)$</td>
<td></td>
<td>2</td>
<td>$\infty$</td>
<td>$\infty$</td>
<td>0</td>
<td>0</td>
<td>$\infty$</td>
</tr>
</tbody>
</table>

**Remark 2.16.** We note that considering the pairs $(L_1, L_2)$ gives the same classification as considering the pairs $(\text{ind}_1, \text{ind}_2)$. To distinguish classes $(T_8)^0$ and $(T_8)^1$ for $c_2 \neq 0$, $c_1 = 0$ we may use Lagrangian tangency order related to component $C_1$. We have $Lt[C_2 : C_1] = 1$ for class $(T_8)^0$ but $Lt[C_2 : C_1] = 2$ for class $(T_8)^1$ if $c_2 \neq 0$, $c_1 = 0$. In similar way, we can distinguish classes $(T_8)^3$ and $(T_8)^4$ for $c_2 \neq 0$, $c_1 = 0$.

**Remark 2.17.** We can see from Table 6 that the Lagrangian tangency order, $Lt$, distinguishes different classes from the index of isotropy, $\text{ind}$. For example, the class $(T_8)^4$ in the case $c_1 = 0$, $c_2 \neq 0$ and the class $(T_8)^6,2$ are distinguished by the index of isotropy, $\text{ind}$, but are not distinguished by the Lagrangian tangency order. We can distinguish these classes using the relative Lagrangian tangency order: for the class $(T_8)^4$ in the case $c_1 = 0$, $c_2 \neq 0$ we have $Lt[C_2 : C_1] = 3$, and for the class $(T_8)^6,2$ we have $Lt[C_2 : C_1] = 4$.

The index of isotropy for the classes $(T_8)^{3,1}$, $(T_8)^4$, $(T_8)^{6,1}$, $(T_8)^{5,1}$ is less than that for the class $(T_8)^{6,2}$ but the analogical inequality does not hold for the Lagrangian tangency order.

We are not able to distinguish all symplectic classes using the Lagrangian tangency orders or the indices of isotropy, but we can do so by checking geometric conditions formulated analogously to the $T_7$ singularity (see §3.2).
Symplectic \( T_7, T_8 \) singularities

Table 6. Lagrangian invariants for symplectic classes of \( T_8 \) singularity when \( \omega|_W = 0 \).

\[ (W \text{ is the tangent space to a non-singular three-dimensional manifold in } (\mathbb{R}^{2n+6}, \omega)\text{ containing } N \in (T_8).) \]

<table>
<thead>
<tr>
<th>class ( (T_8)^{3,1} )</th>
<th>parametrization</th>
<th>conditions</th>
<th>( Lt )</th>
<th>( L_1 )</th>
<th>( L_2 )</th>
<th>ind</th>
<th>ind(_1)</th>
<th>ind(_2)</th>
</tr>
</thead>
<tbody>
<tr>
<td>( (T_8)^{3,1} )</td>
<td>( (t^3, \frac{1}{2}c_2t^4, -t^2, 0, 0, 0, \ldots) )</td>
<td>( c_2 \neq 0 )</td>
<td>3</td>
<td>5</td>
<td>3</td>
<td>1</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>( (T_8)^{3,1} )</td>
<td>( (\pm t^2, \frac{1}{2}c_1t^3, 0, \mp c_1t^3, t, 0, \ldots) )</td>
<td>( c_2 = 0 )</td>
<td>3</td>
<td>( \infty )</td>
<td>3</td>
<td>1</td>
<td>( \infty )</td>
<td>1</td>
</tr>
<tr>
<td>( (T_8)^4 )</td>
<td>( (t^3, \frac{1}{2}c_1t^4, -t^2, 0, 0, 0, \ldots) )</td>
<td>( c_1 \cdot c_2 \neq 0 )</td>
<td>4</td>
<td>5</td>
<td>4</td>
<td>1</td>
<td>1</td>
<td>2</td>
</tr>
<tr>
<td>( (T_8)^4 )</td>
<td>( (\pm t^2, \frac{1}{2}c_2t^3, 0, \mp t^3, t, 0, \ldots) )</td>
<td>( c_1 = 0, c_2 \neq 0 )</td>
<td>4</td>
<td>( \infty )</td>
<td>4</td>
<td>1</td>
<td>( \infty )</td>
<td>2</td>
</tr>
<tr>
<td>( (T_8)^4 )</td>
<td>( (\pm t^2, \frac{1}{2}c_2t^3, 0, \mp t^3, t, 0, \ldots) )</td>
<td>( c_1 \neq 0, c_2 = 0 )</td>
<td>5</td>
<td>5</td>
<td>( \infty )</td>
<td>1</td>
<td>1</td>
<td>( \infty )</td>
</tr>
</tbody>
</table>

\( (T_8)^{6,1} \)
\( (T_8)^{5,1} \)
\( (T_8)^{6,1} \)
\( (T_8)^7 \)
\( (T_8)^8 \)

3. Symplectic \( T_7 \)-singularities

Denote by \( (T_7) \) the class of varieties in a fixed symplectic space \((\mathbb{R}^{2n}, \omega)\) which are diffeomorphic to

\[ T_7 = \{ x \in \mathbb{R}^{2n \geq 4} : x_1^2 + x_2^3 + x_3^3 = x_2x_3 = x_2x_4 = 0 \}. \quad (3.1) \]

This is the classical one-dimensional isolated complete intersection singularity \( T_7 \) \([5, 11]\). \( N \) is quasi-homogeneous with weights \( w(x_1) = 3, w(x_2) = w(x_3) = 2 \).

We use the method of algebraic restrictions to obtain the complete classification of symplectic singularities of \( (T_7) \) presented in the following theorem.

**Theorem 3.1.** Any stratified submanifold of the symplectic space \((\mathbb{R}^{2n}, \sum_{i=1}^n dp_i \wedge dq_i)\) which is diffeomorphic to \( T_7 \) is symplectically equivalent to one and only one of the normal forms \( T_i^d, i = 0, 1, \ldots, 7 \) (respectively, \( i = 0, 1, 2, 4 \)). The parameters \( c, c_1, c_2 \) of the normal forms are moduli:

\[ T_0^d : p_1^2 + p_2^3 + q_2^3 = 0, \; p_2q_2 = 0, \; q_1 = c_1q_2 + c_2p_2, \; p_{\geq 3} = q_{\geq 3} = 0, \; c_1 \cdot c_2 \neq 0; \]
\[ T_1^d : p_1^2 + p_2^3 + q_2^3 = 0, \; p_2q_1 = 0, \; q_2 = c_1q_1 - c_2p_1p_2, \; p_{\geq 3} = q_{\geq 3} = 0; \]
\[ T_2^d : p_1^2 + p_2^3 + q_2^3 = 0, \; p_2q_2 = 0, \; q_1 = \frac{1}{3}c_1q_2^2 + \frac{1}{2}c_2p_2^2, \; p_{\geq 3} = q_{\geq 3} = 0, \; (c_1, c_2) \neq (0, 0); \]
\[ T_4^d : p_1^2 + p_2^3 + q_2^3 = 0, \; p_2q_2 = 0, \; q_1 = \frac{1}{3}c_1q_2^2, \; p_{\geq 3} = q_{\geq 3} = 0; \]
Lagrangian tangency orders and use them to characterize singularities of geometric interpretation (Tables 8 and 9).

eighth and ninth columns of the table.

of Table 7 and the Lagrangian tangency orders of the curve are presented in the seventh, are moduli. The indices of isotropy are presented in the fourth, fifth and sixth columns

668

T

T

T

T

T

ind\((p_1, q_1, \ldots, p_n, q_n)\) is symplectically equivalent to one and only one of the curves presented in the second column of Table 7. The parameters \(c, c_1, c_2\) are moduli. The indices of isotropy are presented in the fourth, fifth and sixth columns of Table 7 and the Lagrangian tangency orders of the curve are presented in the seventh, eighth and ninth columns of the table.
Symplectic $T_7$, $T_8$ singularities

Table 7. The Lagrangian tangency orders and the indices of isotropy for symplectic classes of $T_7$ singularity.

<table>
<thead>
<tr>
<th>class (T)</th>
<th>parametrization of branches</th>
<th>conditions for subclasses</th>
<th>ind</th>
<th>ind_{0}</th>
<th>ind_{t}</th>
<th>$Lt(N)$</th>
<th>$L_{0}$</th>
<th>$L_{1}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$T_{7}^{0}$</td>
<td>$(t^{3}, -c_{2}t^{2}, -t^{2}, 0, 0, \ldots)$</td>
<td>$c_{1} \cdot c_{2} \neq 0$</td>
<td>0</td>
<td>0</td>
<td>2</td>
<td>3</td>
<td>3</td>
<td></td>
</tr>
<tr>
<td>$2n \geq 4$</td>
<td>$(t^{3}, -c_{1}t^{2}, 0, -t^{2}, 0, \ldots)$</td>
<td>$c_{1} \cdot c_{2} \neq 0$</td>
<td>0</td>
<td>0</td>
<td>2</td>
<td>3</td>
<td>3</td>
<td></td>
</tr>
<tr>
<td>$T_{7}^{1}$</td>
<td>$(t^{3}, -t^{2}, 0, -c_{1}t^{2}, 0, \ldots)$</td>
<td>$c_{1} = 0, c_{2} \neq 0$</td>
<td>0</td>
<td>1</td>
<td>3</td>
<td>5</td>
<td>3</td>
<td></td>
</tr>
<tr>
<td>$2n \geq 4$</td>
<td>$(t^{3}, 0, -t^{2}, c_{1}t^{2}, 0, \ldots)$</td>
<td>$c_{1} \neq 0, c_{2} = 0$</td>
<td>0</td>
<td>$\infty$</td>
<td>2</td>
<td>$\infty$</td>
<td>3</td>
<td></td>
</tr>
<tr>
<td>$T_{7}^{2}$</td>
<td>$(t^{3}, \frac{1}{2}c_{1}^{2}t^{2}, 0, -t^{2}, 0, \ldots)$</td>
<td>$c_{1} \cdot c_{2} \neq 0$</td>
<td>0</td>
<td>1</td>
<td>2</td>
<td>5</td>
<td>5</td>
<td></td>
</tr>
<tr>
<td>$2n \geq 4$</td>
<td>$(t^{3}, \frac{1}{2}c_{2}^{2}t^{2}, -t^{2}, 0, 0, \ldots)$</td>
<td>$c_{1} \cdot c_{2} = 0$</td>
<td>0</td>
<td>$\infty$</td>
<td>2</td>
<td>$\infty$</td>
<td>5</td>
<td></td>
</tr>
<tr>
<td>$T_{7}^{3}$</td>
<td>$(t^{3}, \frac{1}{2}c_{1}t^{2}, 0, c_{2}t^{2}, -t^{2}, 0, \ldots)$</td>
<td>$c_{1} \neq 0$</td>
<td>1</td>
<td>1</td>
<td>5</td>
<td>5</td>
<td>5</td>
<td></td>
</tr>
<tr>
<td>$2n \geq 6$</td>
<td>$(t^{3}, \frac{1}{2}c_{1}t^{2}, -t^{2}, 0, 0, 0, \ldots)$</td>
<td>$c_{1} = 0$</td>
<td>1</td>
<td>$\infty$</td>
<td>5</td>
<td>$\infty$</td>
<td>5</td>
<td></td>
</tr>
<tr>
<td>$T_{7}^{4}$</td>
<td>$(t^{3}, \frac{1}{2}c_{2}t^{2}, 0, -t^{2}, 0, \ldots)$</td>
<td>$c_{2} = 0$</td>
<td>0</td>
<td>$\infty$</td>
<td>2</td>
<td>$\infty$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$2n \geq 4$</td>
<td>$(t^{3}, 0, -t^{2}, 0, 0, \ldots)$</td>
<td>$c_{1} \neq 0$</td>
<td>1</td>
<td>$\infty$</td>
<td>5</td>
<td>$\infty$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$T_{7}^{5}$</td>
<td>$(t^{3}, -t^{2}, 0, c_{1}t^{2}, 0, \ldots)$</td>
<td>$c_{1} \neq 0$</td>
<td>0</td>
<td>$\infty$</td>
<td>5</td>
<td>$\infty$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$2n \geq 6$</td>
<td>$(t^{3}, 0, -t^{2}, 0, 0, \ldots)$</td>
<td>$c_{1} \neq 0$</td>
<td>0</td>
<td>$\infty$</td>
<td>7</td>
<td>$\infty$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$T_{7}^{6}$</td>
<td>$(t^{3}, 0, 0, 0, -t^{2}, 0, \ldots)$</td>
<td>$c_{1} \neq 0$</td>
<td>0</td>
<td>$\infty$</td>
<td>5</td>
<td>$\infty$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$2n \geq 6$</td>
<td>$(t^{3}, 0, 0, 0, 0, \ldots)$</td>
<td>$c_{1} \neq 0$</td>
<td>0</td>
<td>$\infty$</td>
<td>7</td>
<td>$\infty$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$T_{7}^{7}$</td>
<td>$(t^{3}, 0, 0, 0, -t^{2}, 0, \ldots)$</td>
<td>$c_{1} \neq 0$</td>
<td>0</td>
<td>$\infty$</td>
<td>5</td>
<td>$\infty$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$2n \geq 6$</td>
<td>$(t^{3}, 0, -t^{2}, 0, 0, \ldots)$</td>
<td>$c_{1} \neq 0$</td>
<td>0</td>
<td>$\infty$</td>
<td>7</td>
<td>$\infty$</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

The comparison of invariants presented in Table 7 shows that the Lagrangian tangency orders distinguish more symplectic classes than the indices of isotropy. The method of calculating these invariants is described in §4.4.

3.2. Geometric conditions for the classes $(T_7)^i$

The classes $(T_7)^i$ can be distinguished geometrically, without using any local coordinate system.

Let $N \in (T_7)$. Then $N$ is the union of two branches: singular one-dimensional irreducible components diffeomorphic to the $A_2$ singularity. In local coordinates they have the form

$$B_1 = \{x_1^2 + x_3^2 = 0, \ x_2 = x_{3+4} = 0\},$$

$$B_2 = \{x_1^2 + x_3^2 = 0, \ x_{3+3} = 0\}.$$ 

Denote by $\ell_1, \ell_2$ the tangent lines at 0 to the branches $B_1$ and $B_2$, respectively. These lines span a 2-space $P_1$. Let $P_2$ be 2-space tangent at 0 to the branch $B_1$ and $P_3$ be 2-space tangent at 0 to the branch $B_2$. Define the line $\ell_3 = P_2 \cap P_3$. The lines $\ell_1, \ell_2, \ell_3$ span a 3-space $W = W(N)$. Equivalently, $W$ is the tangent space at 0 to some (and then any) non-singular 3-manifold containing $N$. 

The classes \((T_7)^i\) satisfy special conditions in terms of the restriction \(\omega|_W\), where \(\omega\) is the symplectic form. For \(N = T_7 = (3.1)\) it is easy to calculate

\[
\ell_1 = \text{span}(\partial/\partial x_3), \quad \ell_2 = \text{span}(\partial/\partial x_2), \quad \ell_3 = \text{span}(\partial/\partial x_1). \tag{3.2}
\]

3.2.1. Geometric conditions for the class \([0]T_7\)

The geometric distinction of the class \((T_7)^7\) follows from Theorem 4.4: \(N \in (T_7)^7\) if and only if \(N\) is contained in a non-singular Lagrangian submanifold. The following theorem gives a simple way to check the latter condition without using algebraic restrictions.

Given a 2-form \(\sigma\) on a non-singular submanifold \(M\) of \(\mathbb{R}^{2n}\) such that \(\sigma(0) = 0\) and a vector \(v \in T_0 M\), we denote by \(L_v \sigma\) the value at 0 of the Lie derivative of \(\sigma\) along a vector field \(V\) on \(M\) such that \(v = V(0)\). The assumption \(\sigma(0) = 0\) implies that the choice of \(V\) is irrelevant.

**Proposition 3.4.** Let \(N \in (T_7)^i\) be a stratified submanifold of a symplectic space \((\mathbb{R}^{2n}, \omega)\). Let \(M^3\) be any non-singular submanifold containing \(N\) and let \(\sigma\) be the restriction of \(\omega\) to \(TM^3\). Let \(v_i \in \ell_i\) be non-zero vectors. If the symplectic form \(\omega\) has zero algebraic restriction to \(N\), then the following conditions are satisfied:

(I) \(\sigma(0) = 0\);

(II) \(L_{v_i} \sigma(v_i, v_j) = 0\) for \(i, j \in \{1, 2\}\);

(III) \(L_{v_i} \sigma(v_3, v_i) = 0\) for \(i \in \{1, 2\}\);

(IV) \(L_{v_i} \sigma(v_3, v_j) = L_{v_i} \sigma(v_3, v_i)\) for \(i \neq j \in \{1, 2\}\).

**Theorem 3.5.** A stratified submanifold \(N \in (T_7)\) of a symplectic space \((\mathbb{R}^{2n}, \omega)\) belongs to the class \((T_7)^i\) if and only if the couple \((N, \omega)\) satisfies corresponding conditions in the last column of Table 8 or Table 9.

The proofs of the theorems of this section are presented in § 4.5.

4. Proofs

4.1. The method of algebraic restrictions

In this section we present basic facts about the method of algebraic restrictions, which is a very powerful tool for the symplectic classification. The details of the method and proofs of all results of this section can be found in [8].

Given a germ of a non-singular manifold \(M\), denote by \(A^p(M)\) the space of all germs at 0 of differential \(p\)-forms on \(M\). Given a subset \(N \subset M\), introduce the following subspaces of \(A^p(M)\):

\[
A^p_N(M) = \{\omega \in A^p(M) : \omega(x) = 0 \text{ for any } x \in N\};
\]

\[
A^p_0(N, M) = \{\alpha + d\beta : \alpha \in A^p_N(M), \beta \in A^{p-1}_N(M)\}.
\]
Table 8. Geometric interpretation of singularity classes of $T_7$ when $\omega|_W \neq 0$.

($W$ is the tangent space to a non-singular three-dimensional manifold in $(\mathbb{R}^{2n+4}, \omega)$ containing $N \in (T_7)$.)

<table>
<thead>
<tr>
<th>class</th>
<th>normal form</th>
<th>geometric conditions</th>
</tr>
</thead>
<tbody>
<tr>
<td>$(T_7)^0$</td>
<td>$[T_7]^0: [\theta + c_1 \theta_2 + c_2 \theta_3]_{T_7}$</td>
<td>$\omega</td>
</tr>
<tr>
<td>$(T_7)^1$</td>
<td>$[T_7]^1: [\theta + c_1 \theta_2 + c_2 \theta_3]_{T_7}$</td>
<td>$\exists i \neq j \in {1, 2} \omega</td>
</tr>
<tr>
<td>$(T_7)^2$</td>
<td>$[T_7]^2: [\theta + c_1 \theta_2 + c_2 \theta_3]_{T_7}$</td>
<td>$\omega</td>
</tr>
<tr>
<td>$(T_7)^3$</td>
<td>$[T_7]^3: [\theta + c_1 \theta_2 + c_2 \theta_3]_{T_7}$</td>
<td>$\omega</td>
</tr>
<tr>
<td>$(T_7)^4$</td>
<td>$[T_7]^4: [\theta + c_1 \theta_2]_{T_7}$</td>
<td>$\omega</td>
</tr>
</tbody>
</table>

**Definition 4.1.** Let $N$ be the germ of a subset of $M$ and let $\omega \in \mathcal{A}^p(M)$. The algebraic restriction of $\omega$ to $N$ is the equivalence class of $\omega$ in $\mathcal{A}^p(M)$, where the equivalence is as follows: $\omega$ is equivalent to $\tilde{\omega}$ if $\omega - \tilde{\omega} \in \mathcal{A}^p_0(N, M)$.

**Notation.** The algebraic restriction of the germ of a $p$-form $\omega$ on $M$ to the germ of a subset $N \subset M$ will be denoted by $[\omega]_N$. By writing $[\omega]_N = 0$ (or saying that $\omega$ has zero algebraic restriction to $N$), we mean that $[\omega]_N = [0]_N$, i.e. $\omega \in \mathcal{A}^p_0(N, M)$.

**Definition 4.2.** Two algebraic restrictions $[\omega]_N$ and $[\tilde{\omega}]_N$ are called diffeomorphic if there exists the germ of a diffeomorphism $\Phi: \tilde{M} \to M$ such that $\Phi(\tilde{N}) = N$ and $\Phi^*([\omega]_N) = [\tilde{\omega}]_N$.

The method of algebraic restrictions applied to singular quasi-homogeneous subsets is based on the following theorem.
Table 9. Geometric interpretation of singularity classes of $T_7$ when $\omega|_W = 0$.

(W is the tangent space to a non-singular three-dimensional manifold in $(\mathbb{R}^{2n}\geq 6, \omega)$ containing $N \in (T_7)$; (I)–(IV) are the conditions of Proposition 3.4.)

<table>
<thead>
<tr>
<th>class</th>
<th>normal form</th>
<th>geometric conditions</th>
</tr>
</thead>
<tbody>
<tr>
<td>$(T_7)^3$</td>
<td>$[T_7^3]: [\theta_4 + c_1\theta_5 + c_2\theta_6]r_7$</td>
<td>(III) is not satisfied and no branch</td>
</tr>
<tr>
<td></td>
<td>$[T_7^3]: [\theta_4 + c_2\theta_6]r_7$</td>
<td>is contained in a Lagrangian submanifold</td>
</tr>
<tr>
<td>$(T_7)^5$</td>
<td>$[T_7^5]: [\theta_6 + c\theta_7]r_7$</td>
<td>(III) is not satisfied but (II) is not and exactly one branch</td>
</tr>
<tr>
<td></td>
<td></td>
<td>is contained in a Lagrangian submanifold</td>
</tr>
<tr>
<td>$(T_7)^6$</td>
<td>$[T_7^6]: [\theta_7]r_7$</td>
<td>(I)–(IV) are satisfied and branches</td>
</tr>
<tr>
<td></td>
<td></td>
<td>are contained in different Lagrangian submanifolds</td>
</tr>
<tr>
<td>$(T_7)^7$</td>
<td>$[T_7^7]: [0]r_7$</td>
<td>(I)–(IV) are satisfied and $N$ is contained in a Lagrangian submanifold</td>
</tr>
</tbody>
</table>

**Theorem 4.3 (Theorem A in [8])**. Let $N$ be the germ of a quasi-homogeneous subset of $\mathbb{R}^{2n}$. Let $\omega_0, \omega_1$ be germs of symplectic forms on $\mathbb{R}^{2n}$ with the same algebraic restriction to $N$. There exists a local diffeomorphism $\Phi$ such that $\Phi(x) = x$ for any $x \in N$ and $\Phi^*\omega_1 = \omega_0$.

Two germs of quasi-homogeneous subsets $N_1, N_2$ of a fixed symplectic space $(\mathbb{R}^{2n}, \omega)$ are symplectically equivalent if and only if the algebraic restrictions of the symplectic form $\omega$ to $N_1$ and $N_2$ are diffeomorphic.

Theorem 4.3 reduces the problem of symplectic classification of germs of singular quasi-homogeneous subsets to the problem of diffeomorphic classification of algebraic restrictions of the germ of the symplectic form to the germs of singular quasi-homogeneous subsets.

The geometric meaning of zero algebraic restriction is explained by the following theorem.

**Theorem 4.4 (Theorem B in [8])**. The germ of a quasi-homogeneous set $N$ of a symplectic space $(\mathbb{R}^{2n}, \omega)$ is contained in a non-singular Lagrangian submanifold if and only if the symplectic form $\omega$ has zero algebraic restriction to $N$.

The following result shows that the method of algebraic restrictions is very powerful tool in symplectic classification of singular curves.

**Theorem 4.5 (Theorem 2 in [6])**. Let $C$ be the germ of a $K$-analytic curve (for $K = \mathbb{R}$ or $K = \mathbb{C}$). Then the space of algebraic restrictions of germs of closed 2-forms to $C$ is a finite-dimensional vector space.

By a $K$-analytic curve we mean a subset of $K^m$ which is locally diffeomorphic to a one-dimensional (possibly singular) $K$-analytic subvariety of $K^m$. Germs of $C$-analytic parametrized curves can be identified with germs of irreducible $C$-analytic curves.
Table 10. Relations towards calculating \([\Lambda^2(\mathbb{R}^{2n})]_N\) for \(N = T_7\).

<table>
<thead>
<tr>
<th>relations</th>
<th>proof</th>
</tr>
</thead>
<tbody>
<tr>
<td>1 ([x_2 , dx_2 \wedge dx_3]_N = 0)</td>
<td>(4.1) &amp; (dx_2)</td>
</tr>
<tr>
<td>2 ([x_3 , dx_2 \wedge dx_3]_N = 0)</td>
<td>(4.1) &amp; (dx_3)</td>
</tr>
<tr>
<td>3 ([x_3 , dx_1 \wedge dx_2]_N = [x_2 , dx_1 \wedge dx_3]_N)</td>
<td>(4.1) &amp; (dx_1)</td>
</tr>
<tr>
<td>4 ([x_1 , dx_1 \wedge dx_2]_N = 0)</td>
<td>(4.2) &amp; (dx_2) and row 2</td>
</tr>
<tr>
<td>5 ([x_1 , dx_1 \wedge dx_3]_N = 0)</td>
<td>(4.2) &amp; (dx_3) and row 1</td>
</tr>
<tr>
<td>6 ([x_3^2 , dx_1 \wedge dx_2]_N = [x_3^2 , dx_3 \wedge dx_1]_N)</td>
<td>(4.2) &amp; (dx_1)</td>
</tr>
<tr>
<td>7 ([x_3^2 , dx_1 \wedge dx_2]_N = 0)</td>
<td>rows 1 and 2 and ([x_3^2]_N = [-x_2^3 - x_3^3]_N)</td>
</tr>
<tr>
<td>8 ([x_3^2 , dx_1 \wedge dx_3]_N = 0)</td>
<td>(4.1) &amp; (x_3 , dx_1) and ([x_2 x_3]_N = 0)</td>
</tr>
</tbody>
</table>

In the remainder of this paper we use the following notation:

- \([\Lambda^2(\mathbb{R}^{2n})]_N\) is the vector space consisting of algebraic restrictions of germs of all 2-forms on \(\mathbb{R}^{2n}\) to the germ of a subset \(N \subset \mathbb{R}^{2n}\);

- \([Z^2(\mathbb{R}^{2n})]_N\) is the subspace of \([\Lambda^2(\mathbb{R}^{2n})]_N\) consisting of algebraic restrictions of germs of all closed 2-forms on \(\mathbb{R}^{2n}\) to \(N\);

- \([\text{Symp}(\mathbb{R}^{2n})]_N\) is the open set in \([Z^2(\mathbb{R}^{2n})]_N\) consisting of algebraic restrictions of germs of all symplectic 2-forms on \(\mathbb{R}^{2n}\) to \(N\).

For calculating discrete invariants we use the following propositions.

**Proposition 4.6 (Domitrz et al. [8]).** The symplectic multiplicity of the germ of a quasi-homogeneous subset \(N\) in a symplectic space is equal to the codimension of the orbit of the algebraic restriction \([\omega]_N\) with respect to the group of local diffeomorphisms preserving \(N\) in the space of algebraic restrictions of closed 2-forms to \(N\).

**Proposition 4.7 (Domitrz et al. [8]).** The index of isotropy of the germ of a quasi-homogeneous subset \(N\) in a symplectic space \((\mathbb{R}^{2n}, \omega)\) is equal to the maximal order of vanishing of closed 2-forms representing the algebraic restriction \([\omega]_N\).

**Proposition 4.8 (Domitrz [6]).** Let \(f\) be the germ of a quasi-homogeneous curve such that the algebraic restriction of a symplectic form to it can be represented by a closed 2-form vanishing at 0. Then the Lagrangian tangency order of the germ of a quasi-homogeneous curve \(f\) is the maximum of the order of vanishing on \(f\) over all 1-forms \(\alpha\) such that \([\omega]_f = [d\alpha]_f\).

### 4.2. Algebraic restrictions to \(T_7\) and their classification

One has the following relations for \((T_7)\)-singularities:

\[
[d(x_2 x_3)]_{T_7} = [x_2 \, dx_3 + x_3 \, dx_2]_{T_7} = 0,
\]

\[
[d(x_2^2 + x_3^2)]_{T_7} = [2x_1 \, dx_1 + 3x_2^2 \, dx_2 + 3x_3^2 \, dx_3]_{T_7} = 0.
\]

Multiplying these relations by suitable 1-forms, we obtain the relations in Table 10.
Using the method of algebraic restrictions and Table 10, we obtain the following proposition.

**Proposition 4.9.** \([A^2(\mathbb{R}^{2n})]_{T_7}\) is an eight-dimensional vector space spanned by the algebraic restrictions to \(T_7\) of the 2-forms:

\[
\begin{align*}
\theta_1 &= dx_2 \wedge dx_3, & \theta_2 &= dx_1 \wedge dx_3, & \theta_3 &= dx_1 \wedge dx_2, \\
\theta_4 &= x_3 dx_1 \wedge dx_3, & \theta_5 &= x_2 dx_1 \wedge dx_2, \\
\sigma_1 &= x_3 dx_1 \wedge dx_2, & \sigma_2 &= x_1 dx_2 \wedge dx_3, \\
\theta_7 &= x_2^2 dx_1 \wedge dx_3.
\end{align*}
\]

Proposition 4.9 and the results of §4.1 imply the following description of the space \([Z^2(\mathbb{R}^{2n})]_{T_7}\) and the manifold \([\text{Symp}(\mathbb{R}^{2n})]_{T_7}\).

**Theorem 4.10.** \([Z^2(\mathbb{R}^{2n})]_{T_7}\) is a seven-dimensional vector space spanned by the algebraic restrictions to \(T_7\) of the quasi-homogeneous 2-forms \(\theta_i\):

\[
\theta_1, \theta_2, \theta_3, \theta_4, \theta_5, \theta_6 = \sigma_1 - \sigma_2, \theta_7.
\]

If \(n \geq 3\), then \([\text{Symp}(\mathbb{R}^{2n})]_{T_7}\) is an open part of the 7-space \([Z^2(\mathbb{R}^4)]_{T_7}\) consisting of algebraic restrictions of the form \(c_1 \theta_1 + \cdots + c_7 \theta_7\) such that \((c_1, c_2, c_3) \neq (0, 0, 0)\).

**Theorem 4.11.**

(i) Any algebraic restriction in \([Z^2(\mathbb{R}^{2n})]_{T_7}\) can be brought by a symmetry of \(T_7\) to one of the normal forms \([T_7]^i\) given in the second column of Table 11.

(ii) The codimension in \([Z^2(\mathbb{R}^{2n})]_{T_7}\) of the singularity class corresponding to the normal form \([T_7]^i\) is equal to \(i\).

(iii) The singularity classes corresponding to the normal forms are disjoint.

(iv) The parameters \(c, c_1, c_2\) of the normal forms \([T_7]^0, [T_7]^1, [T_7]^2, [T_7]^3, [T_7]^4, [T_7]^5\) are moduli.

The proof of Theorem 4.11 is presented in §4.6.

In the first column of Table 11, we denote by \((T_7)^i\) a subclass of \((T_7)\) consisting of \(N \in (T_7)\) such that the algebraic restriction \([\omega]_N\) is diffeomorphic to some algebraic restriction of the normal form \([T_7]^i\). Theorems 4.3 and 4.11 and Proposition 4.10 imply the following statement, which explains why the given stratification of \((T_7)\) is natural.

**Theorem 4.12.** Fix \(i \in \{0, 1, \ldots, 7\}\). All stratified submanifolds \(N \in (T_7)^i\) have the same

(a) symplectic multiplicity and

(b) index of isotropy given in Table 11 by \((T_7)^i\).
Table 11. Classification of symplectic $T_7$ singularities.

(‘cod’ denotes the codimension of the classes; $\mu^{sym}$ denotes the symplectic multiplicity; ‘ind’ denotes the index of isotropy.)

<table>
<thead>
<tr>
<th>symplectic class</th>
<th>normal forms for algebraic restrictions</th>
<th>cod</th>
<th>$\mu^{sym}$</th>
<th>ind</th>
</tr>
</thead>
<tbody>
<tr>
<td>$(T_7)^0$ $(2n \geq 4)$</td>
<td>$[T_7]^0$: $[\theta_1 + c_1 \theta_2 + c_2 \theta_3, c_1 \cdot c_2 \neq 0]$</td>
<td>0</td>
<td>2</td>
<td>0</td>
</tr>
<tr>
<td>$(T_7)^1$ $(2n \geq 4)$</td>
<td>$[T_7]^1$: $[c_1 \theta_1 + \theta_2 + c_2 \theta_3]_7$, $c_1 \cdot c_2 \neq 0$</td>
<td>1</td>
<td>3</td>
<td>0</td>
</tr>
<tr>
<td>$(T_7)^2$ $(2n \geq 4)$</td>
<td>$[T_7]^2$: $[\theta_1 + c_1 \theta_2 + c_2 \theta_3]_7$, $(c_1, c_2) \neq (0, 0)$</td>
<td>2</td>
<td>4</td>
<td>0</td>
</tr>
<tr>
<td>$(T_7)^3$ $(2n \geq 6)$</td>
<td>$[T_7]^3$: $[\theta_1 + c_2 \theta_3 + c_3 \theta_4]_7$, $(c_1, c_2) \neq (0, 0)$</td>
<td>3</td>
<td>5</td>
<td>1</td>
</tr>
<tr>
<td>$(T_7)^4$ $(2n \geq 4)$</td>
<td>$[T_7]^4$: $[\theta_1 + c \theta_2]_7$, $c_1 \cdot c_2 \neq 0$</td>
<td>4</td>
<td>5</td>
<td>0</td>
</tr>
<tr>
<td>$(T_7)^5$ $(2n \geq 6)$</td>
<td>$[T_7]^5$: $[\theta_6 + c \theta_7]_7$, $(c_1, c_2) \neq (0, 0)$</td>
<td>5</td>
<td>6</td>
<td>1</td>
</tr>
<tr>
<td>$(T_7)^6$ $(2n \geq 6)$</td>
<td>$[T_7]^6$: $[\theta_7]_7$, $(c_1, c_2) \neq (0, 0)$</td>
<td>6</td>
<td>6</td>
<td>2</td>
</tr>
<tr>
<td>$(T_7)^7$ $(2n \geq 6)$</td>
<td>$[T_7]^7$: $[0]_7$, $(c_1, c_2) \neq (0, 0)$</td>
<td>7</td>
<td>7</td>
<td>$\infty$</td>
</tr>
</tbody>
</table>

Proof. Part (a) follows from Proposition 4.6 and Theorem 4.11 and the fact that the codimension in $[Z^2(\mathbb{R}^{2n})]_{T_7}$ of the orbit of an algebraic restriction $a \in [T_7]^i$ is equal to the sum of the number of moduli in the normal form $[T_7]^i$ and the codimension in $[Z^2(\mathbb{R}^{2n})]_{T_7}$ of the class of algebraic restrictions defined by this normal form.

Part (b) follows from Theorem 4.4 and Proposition 4.7.

Proposition 4.13. The classes $(T_7)^i$ are symplectic singularity classes, i.e. they are closed with respect to the action of the group of symplectomorphisms. The class $(T_7)$ is the disjoint union of the classes $(T_7)^i$, $i \in \{0, 1, \ldots, 7\}$. The classes $(T_7)^0$, $(T_7)^1$, $(T_7)^2$, $(T_7)^4$ are non-empty for any dimension $2n \geq 4$ of the symplectic space; the classes $(T_7)^3$, $(T_7)^5$, $(T_7)^6$, $(T_7)^7$ are empty if $n = 2$ and not empty if $n \geq 3$.

4.3. Symplectic normal forms

Let us transfer the normal forms $[T_7]^i$ to symplectic normal forms. Fix a family $\omega^i$ of symplectic forms on $\mathbb{R}^{2n}$ realizing the family $[T_7]^i$ of algebraic restrictions. We can fix, for example,

$\omega^0 = \theta_1 + c_1 \theta_2 + c_2 \theta_3 + dx_1 \wedge dx_4 + dx_5 \wedge dx_6 + \cdots + dx_{2n-1} \wedge dx_{2n}$, $c_1 \cdot c_2 \neq 0,$

$\omega^1 = \theta_1 + \theta_2 + c_2 \theta_5 + dx_2 \wedge dx_4 + dx_5 \wedge dx_6 + \cdots + dx_{2n-1} \wedge dx_{2n}$,

$\omega^2 = \theta_1 + c_1 \theta_4 + c_2 \theta_5 + dx_1 \wedge dx_3 + dx_5 \wedge dx_6 + \cdots + dx_{2n-1} \wedge dx_{2n}$, $(c_1, c_2) \neq (0, 0),$  

$\omega^3 = \theta_4 + c_1 \theta_5 + c_2 \theta_6 + dx_1 \wedge dx_3 + dx_2 \wedge dx_5 + dx_3 \wedge dx_6$  

$+ dx_7 \wedge dx_8 + \cdots + dx_{2n-1} \wedge dx_{2n},$

$\omega^4 = \theta_1 + c_2 \theta_7 + dx_1 \wedge dx_4 + dx_5 \wedge dx_6 + \cdots + dx_{2n-1} \wedge dx_{2n},$

$\omega^5 = \theta_6 + c \theta_7 + dx_1 \wedge dx_4 + dx_2 \wedge dx_5 + dx_3 \wedge dx_6 + dx_7 \wedge dx_8 + \cdots + dx_{2n-1} \wedge dx_{2n},$

$\omega^6 = \theta_7 + dx_1 \wedge dx_4 + dx_2 \wedge dx_5 + dx_3 \wedge dx_6 + dx_7 \wedge dx_8 + \cdots + dx_{2n-1} \wedge dx_{2n},$

$\omega^7 = dx_1 \wedge dx_4 + dx_2 \wedge dx_5 + dx_3 \wedge dx_6 + dx_7 \wedge dx_8 + \cdots + dx_{2n-1} \wedge dx_{2n}.$
Let \( \omega = \sum_{i=1}^{m} dp_i \wedge dq_i \), where \((p_1, q_1, \ldots, p_n, q_n)\) is the coordinate system on \( \mathbb{R}^{2n}, n \geq 3 \) (respectively, \( n = 2 \)). Fix, for \( i = 0, 1, \ldots, 7 \) (respectively, for \( i = 0, 1, 2, 4 \)) a family \( \Phi^i \) of local diffeomorphisms which bring the family of symplectic forms \( \omega^i \) to the symplectic form \( \omega: (\Phi^i)^* \omega^i = \omega \). Consider the families \( T^i_2 = (\Phi^i)^{-1}(T_7) \). Any stratified submanifold of the symplectic space \((\mathbb{R}^{2n}, \omega)\) which is diffeomorphic to \( T_7 \) is symplectically equivalent to one and only one of the normal forms \( T^i_7, i = 0, 1, \ldots, 7 \) (respectively, \( i = 0, 1, 2, 4 \)) presented in Theorem 3.1. By Theorem 4.11 we obtain that parameters \( c, c_1, c_2 \) of the normal forms are moduli.

### 4.4. Proof of Theorem 3.3

The numbers \( \text{ind}(B_1) \) and \( \text{ind}(B_2) \) are computed using Proposition 4.7 for branches \( B_1 \) and \( B_2 \). The space \([Z^2(\mathbb{R}^{2n})]_{B_1}\) is spanned only by the algebraic restrictions to \( B_1 \) of the 2-forms \( \theta_2, \theta_4 \). The space \([Z^2(\mathbb{R}^{2n})]_{B_2}\) is spanned only by the algebraic restrictions to \( B_2 \) of the 2-forms \( \theta_3, \theta_5 \). Branches are curves of type \( A_2 \), and from Table 1 we know the interaction between the index of isotropy and the Lagrangian tangency order. Knowing \( \text{ind}(B_1) \) and \( \text{ind}(B_2) \), we obtain \( L_t(B_1) = 3 + \text{ind}(B_1) \) and \( L_t(B_2) = 3 + \text{ind}(B_2) \). Then \( L_t \) is the minimum of these numbers and \( L_n \) is their maximum. Next we calculate \( L_t(N) \) by definition, finding the nearest Lagrangian submanifold to the branches, knowing that it cannot be greater than \( L_t \).

As an example we calculate the invariants for the class \((T_7)^1\).

We have \([\omega^1]_{B_1} = [c_1 \theta_1 + \theta_2 + c_2 \theta_5]_{B_1} = [\theta_2]_{B_1}\) and thus \( \text{ind}(B_1) = 0 \) and \( L_t(B_1) = 3 \).

\([\omega^1]_{B_2} = [c_1 \theta_1 + \theta_2 + c_2 \theta_5]_{B_2} = [c_2 \theta_5]_{B_2}\) and thus \( \text{ind}(B_2) = 1 \) and \( L_t(B_2) = 5 \) if \( c_2 \neq 0 \) and \( \text{ind}(B_2) = \infty \) and \( L_t(B_2) = \infty \) if \( c_2 = 0 \).

Finally, for the class \((T_7)^1\) we have \( L_n = 5 \) if \( c_2 \neq 0 \) and \( L_n = \infty \) if \( c_2 = 0 \) and \( L_t = 3 \) so \( L_t(N) \leq 3 \).

For the smooth Lagrangian submanifolds \( L \) defined by the conditions \( p_1 = 0, q_2 = 0 \) and \( p_i = 0, i > 2 \), we get \( t[N, L] = 3 \) if \( c_1 = 0 \); thus, \( L_t(N) = 3 \) in this case. But, if \( c_1 \neq 0 \), then \( t[N, L] = 2 \) and it cannot be greater for any other smooth Lagrangian submanifold, so \( L_t(N) = 2 \) in this case.

### 4.5. Proof of Theorem 3.5

**Proof of Proposition 3.4.** Any 2-form \( \sigma \) which has zero algebraic restriction to \( T_7 \) can be expressed in the following form:

\[
\sigma = H_1 \alpha + H_2 \beta + dH_1 \wedge \gamma + dH_2 \wedge \delta,
\]

where \( H_1 = x_1^2 + x_2^3 + x_3^3 \), \( H_2 = x_2 x_3 \) and \( \alpha, \beta \) are 2-forms on \( TM^3 \) and \( \gamma = \gamma_1 dx_1 + \gamma_2 dx_2 + \gamma_3 dx_3 \) and \( \delta = \delta_1 dx_1 + \delta_2 dx_2 + \delta_3 dx_3 \) are 1-forms on \( TM^3 \). Since

\[
H_1(0) = H_2(0) = 0, \quad dH_1|_0 = dH_2|_0 = 0,
\]

we obtain the following equality:

\[
L_{\sigma} \sigma = d(V \mid \sigma)|_0 + (V \mid d\sigma)|_0 = d(V \mid \sigma)|_0.
\]
Symplectic $T_7, T_8$ singularities

Equation (4.3) also implies that
\[ d(V \mid \sigma)|_0 = d(V \mid dH_1)|_0 \wedge \gamma|_0 + d(V \mid dH_2)|_0 \wedge \delta|_0. \]

By simple calculation we get
\[
\begin{align*}
\mathcal{L}_{v_1}\sigma &= dx_2 \wedge \delta|_0 = \delta_3|_0 dx_2 \wedge dx_3 - \delta_1|_0 dx_1 \wedge dx_2, \\
\mathcal{L}_{v_2}\sigma &= dx_3 \wedge \delta|_0 = \delta_1|_0 dx_3 \wedge dx_1 - \delta_2|_0 dx_2 \wedge dx_3, \\
\mathcal{L}_{v_3}\sigma &= 2 dx_1 \wedge \gamma|_0 = 2\gamma_2|_0 dx_1 \wedge dx_2 - 2\gamma_3|_0 dx_3 \wedge dx_1.
\end{align*}
\]

Finally, we obtain
\[
\begin{align*}
\mathcal{L}_{v_1}\sigma(v_3, v_1) &= 0, & \mathcal{L}_{v_2}\sigma(v_3, v_2) &= 0, & \mathcal{L}_{v_3}\sigma(v_1, v_2) &= 0, \\
\mathcal{L}_{v_1}\sigma(v_3, v_2) &= -\delta_1|_0 = \mathcal{L}_{v_2}\sigma(v_3, v_1).
\end{align*}
\]

**Proof of Theorem 3.5.** The conditions on the pair $(\omega, N)$ in the last columns of Tables 8 and 9 are disjoint. It suffices to prove that these conditions in the row of $(T_7)^i$ are satisfied for any $N \in (T_7)^i$. This is a corollary of the following claims.

1. Each of the conditions in the last column of Tables 8 and 9 is invariant with respect to the action of the group of diffeomorphisms in the space of pairs $(\omega, N)$.

2. Each of these conditions depends only on the algebraic restriction $[\omega]|_N$.

3. Take the simplest 2-forms $\omega^i$ representing the normal forms $[T_7]^i$ for algebraic restrictions: $\omega^0, \omega^1, \omega^2, \omega^3, \omega^4, \omega^5, \omega^6, \omega^7$. The pair $(\omega = \omega^i, T_7)$ satisfies the condition in the last column of Table 8 or Table 9 (the row of $(T_7)^i$).

To prove the third statement we note that in the case $N = T_7 = (3.1)$ one has
\[ W = \text{span}(\partial/\partial x_1, \partial/\partial x_2, \partial/\partial x_3) \]
and
\[
\begin{align*}
v_1 &\in \ell_1 = \text{span}(\partial/\partial x_3), \\
v_2 &\in \ell_2 = \text{span}(\partial/\partial x_2), \\
v_3 &\in \ell_3 = \text{span}(\partial/\partial x_1).
\end{align*}
\]

By simple calculation and observation of Lagrangian tangency orders, we obtain that following statements are true.

$(T^0)$ $\omega^0|_{\ell_1 + \ell_2} \neq 0, \omega^0|_{\ell_1 + \ell_3} \neq 0, \omega^0|_{\ell_2 + \ell_3} \neq 0, L_m < \infty$ and $L_t < \infty$; hence, no branch is contained in a smooth Lagrangian submanifold.
4.6. Proof of Theorem 4.11

For any $c_1$ and $c_2$, $\omega^1|_{t_1+\varepsilon_3} = 0$ and $\omega^1|_{t_2+\varepsilon_3} \neq 0$ or $\omega^1|_{t_1+\varepsilon_3} \neq 0$ and $\omega^1|_{t_2+\varepsilon_3} = 0$. If $c_2 = 0$, then $L_n = \infty$ and $L_\ell < \infty$; hence, exactly one branch is contained in some smooth Lagrangian submanifold. For $c_2 \neq 0$, $L_n < \infty$ and $L_\ell < \infty$, so no branch is contained in a smooth Lagrangian submanifold. $\omega^1|_{t_1+\varepsilon_2} = 0$ if and only if $c_1 = 0$.

For any $c_1$ and $c_2$, $\omega^2|_{t_1+\varepsilon_3} \neq 0$, $\omega^2|_{t_2+\varepsilon_3} = 0$ and $\omega^2|_{t_2+\varepsilon_3} = 0$. If $c_1 \cdot c_2 \neq 0$, then $L_n < \infty$ and $L_\ell < \infty$ so no branch is contained in a Lagrangian submanifold. If $c_1 = 0$ and $c_2 \neq 0$ or $c_1 \neq 0$ and $c_2 = 0$, then $L_n = \infty$ and $L_\ell < \infty$; hence, exactly one branch is contained in some smooth Lagrangian submanifold.

The Lie derivative of $\omega^3 = \theta_4 + c_1\theta_5 + c_2\theta_6$ along a vector field $V = \partial/\partial x_3$ is not equal to 0, so condition (III) of Proposition 3.4 is not satisfied. If $c_1 \neq 0$, then $L_n < \infty$ and $L_\ell < \infty$; hence, no branch is contained in a Lagrangian submanifold. If $c_1 = 0$, then $L_n = \infty$ and $L_\ell < \infty$; hence, only one branch is contained in some Lagrangian submanifold.

For any $c$, $\omega^4|_{t_1+\varepsilon_3} \neq 0$, $\omega^4|_{t_2+\varepsilon_3} = 0$ and $\omega^4|_{t_2+\varepsilon_3} = 0$. Both branches are contained in different Lagrangian submanifolds since $L_n = L_\ell = \infty$ and $Lt(N) < \infty$.

We can calculate the Lie derivatives of $\omega^5 = \theta_6 + c\theta_7$ along vector fields $V_1 = \partial/\partial x_3$, $V_2 = \partial/\partial x_2$ and $V_3 = \partial/\partial x_3$: $\mathcal{L}_{V_1}\omega^5(V_3, V_1) = 0$ and $\mathcal{L}_{V_2}\omega^5(V_3, V_2) = 0$, so condition (III) of Proposition 3.4 is satisfied, but the Lie derivative $\mathcal{L}_{V_3}\omega^5(V_1, V_2)$ is not equal to 0, so condition (II) of Proposition 3.4 is not satisfied. We have $Lt(N) < \infty$ and $L_n = L_\ell = \infty$; hence, branches are contained in different Lagrangian submanifolds.

The Lie derivatives of $\omega^6 = \theta_7$, $\mathcal{L}_{V_i}\omega^6(V_j, V_k) = 0$ for $i, j, k \in \{1, 2, 3\}$, so conditions (II)–(IV) of Proposition 3.4 are satisfied. We have $Lt(N) < \infty$ and $L_n = L_\ell = \infty$; hence, branches are contained in different Lagrangian submanifolds.

For $\omega^7 = 0$ we have $\mathcal{L}_{V_i}\omega^7(V_j, V_k) = 0$ for $i, j, k \in \{1, 2, 3\}$, so conditions (II)–(IV) of Proposition 3.4 are satisfied. The condition $Lt(N) = \infty$ implies the curve $N$ is contained in a smooth Lagrangian submanifold.

4.6. Proof of Theorem 4.11

In our proof we use vector fields tangent to $N \in (T_7)$. A Hamiltonian vector field is an example of such a vector field. We recall by [4] a suitable definition and facts.

**Definition 4.14.** Let $H = \{H_1 = \cdots = H_p = 0\} \subset \mathbb{R}^n$ be a complete intersection. Consider a set of $p + 1$ integers $1 \leq i_1 < \cdots < i_{p+1} \leq n$. A Hamiltonian vector field $X_H(i_1, \ldots, i_{p+1})$ on a complete intersection $H$ is the determinant obtained by expansion
Table 12. Infinitesimal actions on algebraic restrictions of closed 2-forms to $T_7$.

($E$ is defined as in (4.5).)

<table>
<thead>
<tr>
<th></th>
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<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>$X_1 = x_3E$</td>
<td>$0$</td>
<td>$7[\theta_4]$</td>
<td>$3[\theta_6]$</td>
<td>$9[\theta_7]$</td>
<td>$0$</td>
<td>$0$</td>
<td>$0$</td>
</tr>
<tr>
<td>$X_2 = x_2E$</td>
<td>$0$</td>
<td>$-3[\theta_6]$</td>
<td>$7[\theta_5]$</td>
<td>$0$</td>
<td>$-9[\theta_7]$</td>
<td>$0$</td>
<td>$0$</td>
</tr>
<tr>
<td>$X_3 = x_1E$</td>
<td>$-4[\theta_6]$</td>
<td>$0$</td>
<td>$0$</td>
<td>$0$</td>
<td>$0$</td>
<td>$0$</td>
<td>$0$</td>
</tr>
<tr>
<td>$X_4 = x_2^2E$</td>
<td>$0$</td>
<td>$0$</td>
<td>$-9[\theta_7]$</td>
<td>$0$</td>
<td>$0$</td>
<td>$0$</td>
<td>$0$</td>
</tr>
<tr>
<td>$X_5 = x_3^2E$</td>
<td>$0$</td>
<td>$9[\theta_7]$</td>
<td>$0$</td>
<td>$0$</td>
<td>$0$</td>
<td>$0$</td>
<td>$0$</td>
</tr>
</tbody>
</table>

with respect to the first row of the symbolic $(p + 1) \times (p + 1)$ matrix

$$X_H(i_1, \ldots, i_{p+1}) = \det \begin{bmatrix}
\frac{\partial}{\partial x_{i_1}} & \cdots & \frac{\partial}{\partial x_{i_{p+1}}} \\
\frac{\partial H_1}{\partial x_{i_1}} & \cdots & \frac{\partial H_1}{\partial x_{i_{p+1}}} \\
\vdots & \ddots & \vdots \\
\frac{\partial H_p}{\partial x_{i_1}} & \cdots & \frac{\partial H_p}{\partial x_{i_{p+1}}}
\end{bmatrix}. \quad (4.4)$$

**Theorem 4.15 (Wahl [15]).** Let $H = \{H_1 = \cdots = H_p = 0\} \subset \mathbb{R}^n$ be a positive-dimensional complete intersection with an isolated singularity. If $H_1, \ldots, H_p$ are quasi-homogeneous with positive weights $\lambda_1, \ldots, \lambda_n$, then the module of vector fields tangent to $H$ is generated by the Euler vector field

$$E = \sum_{i=1}^n \lambda_i x_i \frac{\partial}{\partial x_i}$$

and the Hamiltonian fields $X_H(i_1, \ldots, i_{p+1})$, where the numbers $i_1, \ldots, i_{p+1}$ run through all possible sets $1 \leq i_1 < \cdots < i_{p+1} \leq n$.

**Proposition 4.16.** Let $H = \{H_1 = \cdots = H_{n-1} = 0\} \subset \mathbb{R}^n$ be a one-dimensional complete intersection. If $X_H$ is the Hamiltonian vector field on $H$, then $[\mathcal{L}_{X_H}(\alpha)]_H = [0]_H$ for any closed 2-form $\alpha$.

**Proof.** Note that $X_H \mid dx_1 \wedge \cdots \wedge dx_n = dH_1 \wedge \cdots \wedge dH_{n-1}$. This implies that, for $i < j$,

$$X_H \mid dx_i \wedge dx_j = (-1)^{i+j+1} \left( \frac{\partial}{\partial x_{i_1}} \wedge \cdots \wedge \frac{\partial}{\partial x_{i_{n-2}}} \right) \left( dH_1 \wedge \cdots \wedge dH_{n-1} \right)$$

$$= \sum_{k=1}^{n-1} (-1)^{k+i+j} \left( \frac{\partial}{\partial x_{i_1}} \wedge \cdots \wedge \frac{\partial}{\partial x_{i_{n-2}}} \right) \left( dH_{i_1,k} \wedge \cdots \wedge dH_{i_{n-2},k} \right) dH_k$$

$$= \sum_{k=1}^{n-1} f_k dH_k,$$
where \((i_1, \ldots, i_{n-2}) = (1, \ldots, i-1, i+1, \ldots, j-1, j+1, \ldots, n)\), and for \(k \in \{1, \ldots, n-1\}\) we take a sequence \((t_{1,k}, \ldots, t_{n-2,k}) = (1, \ldots, k-1, k+1, \ldots, n-1)\).

Thus,

\[
[X_H \mid dx_i \wedge dx_j]_{H=0} = \left[ \sum_{k=1}^{n-1} f_k \, dH_k \right]_{H} = [0].
\]

If

\[
\alpha = \sum_{i<j} g_{i,j} \, dx_i \wedge dx_j
\]

is a closed 2-form, then \([\mathcal{L}_{X_H} \alpha]_{H} = [d(X_H \mid \alpha)]_{H}\). It implies that

\[
[\mathcal{L}_{X_H} \alpha]_{H} = \sum_{i<j} g_{i,j} [d(X_H \mid dx_i \wedge dx_j)]_{H} + [d g_{i,j} \wedge (X_H \mid dx_i \wedge dx_j)]_{H} = [0].
\]

\[\square\]

The germ of a vector field tangent to \(T_7\) of non-trivial action on algebraic restriction of closed 2-forms to \(T_7\) may be described as a linear combination of germs of vector fields:

\[X_0 = E, \ X_1 = x_3E, \ X_2 = x_2E, \ X_3 = x_1E, \ X_4 = x_2^2E, \ X_5 = x_3^2E, \text{ where } E \text{ is the Euler vector field} \]

\[
E = 3x_1 \frac{\partial}{\partial x_1} + 2x_2 \frac{\partial}{\partial x_2} + 2x_3 \frac{\partial}{\partial x_3}.
\]  

**Proposition 4.17.** The infinitesimal action of germs of quasi-homogeneous vector fields tangent to \(N \in (T_7)\) on the basis of the vector space of algebraic restrictions of closed 2-forms to \(N\) is presented in Table 12.

Let \(A = [c_1 \theta_1 + c_2 \theta_2 + c_3 \theta_3 + c_4 \theta_4 + c_5 \theta_5 + c_6 \theta_6 + c_7 \theta_7]_{T_7}\) be the algebraic restriction of a symplectic form \(\omega\).

The first statement of Theorem 4.11 follows from the following lemmas.

**Lemma 4.18.** If \(c_1 \cdot c_2 \cdot c_3 \neq 0\), then the algebraic restriction \(A = \sum_{k=1}^{7} c_k \theta_k\) can be reduced by a symmetry of \(T_7\) to an algebraic restriction \([\theta_1 + c_2 \theta_2 + c_3 \theta_3]_{T_7}\).

**Proof of Lemma 4.18.** We use the homotopy method to prove that \(A\) is diffeomorphic to \([\theta_1 + c_2 \theta_2 + c_3 \theta_3]_{T_7}\).

Let

\[
B_t = [c_1 \theta_1 + c_2 \theta_2 + c_3 \theta_3 + (1-t)c_4 \theta_4 + (1-t)c_5 \theta_5 + (1-t)c_6 \theta_6 + (1-t)c_7 \theta_7]_{T_7}
\]

for \(t \in [0; 1]\). Then \(B_0 = A\) and \(B_1 = [c_1 \theta_1 + c_2 \theta_2 + c_3 \theta_3]_{T_7}\). We prove that there exists a family \(\Phi_t \in \text{Symm}(T_7), t \in [0; 1]\) such that

\[
\Phi_t^* B_t = B_0, \quad \Phi_0 = \text{id}.
\]  

Let \(V_t\) be a vector field defined by \(d\Phi_t/dt = V_t(\Phi_t)\). Then, by differentiating (4.6), we obtain

\[
\mathcal{L}_{V_t} B_t = c_4 \theta_4 + c_5 \theta_5 + c_6 \theta_6 + c_7 \theta_7. 
\]  

(4.7)
We are looking for $V_t$ in the form

$$V_t = \sum_{k=1}^{5} b_k(t)X_k,$$

where $b_k(t)$ for $k = 1, \ldots, 5$ are smooth functions $b_k : [0; 1] \to \mathbb{R}$. Then, by Proposition 4.17, (4.7) has the form

$$\begin{bmatrix}
7c_2 & 0 & 0 & 0 & 0 \\
0 & 7c_3 & 0 & 0 & 0 \\
3c_4 & -3c_2 & -4c_1 & 0 & 0 \\
9c_4(1-t) & -9c_5(1-t) & 0 & -9c_3 & 9c_2
\end{bmatrix}
\begin{bmatrix}
b_1 \\
b_2 \\
b_3 \\
b_4 \\
b_5
\end{bmatrix}
= 
\begin{bmatrix}
c_4 \\
c_5 \\
c_6 \\
c_7
\end{bmatrix}. \tag{4.8}$$

If $c_1 \cdot c_2 \cdot c_3 \neq 0$, we can solve (4.8), and $\Phi_t$ may be obtained as a flow of vector field $V_t$. The family $\Phi_t$ preserves $T_7$, because $V_t$ is tangent to $T_7$ and $\Phi_t^\ast B_t = A$. Using the homotopy arguments, we have that $A$ is diffeomorphic to $B_1 = [c_1 \theta_1 + c_2 \theta_2 + c_3 \theta_3]_{T_7}$. By the condition $c_1 \neq 0$ we have a diffeomorphism $\Psi \in \text{Symm}(T_7)$ of the form

$$\Psi : (x_1, x_2, x_3) \mapsto (|c_1|^{-3/4} x_1, |c_1|^{-1/2} x_2, |c_1|^{-1/2} x_3) \tag{4.9}$$

and we obtain

$$\Psi^\ast (B_1) = \left[ \frac{c_1}{|c_1|} \theta_1 + c_2 |c_1|^{-5/4} \theta_2 + c_3 |c_1|^{-5/4} \theta_3 \right]_{T_7} = [\pm \theta_1 + \tilde{c}_2 \theta_2 + \tilde{c}_3 \theta_3]_{T_7}.$$

By the symmetry of $T_7 : (x_1, x_2, x_3) \mapsto (x_1, x_3, x_2)$, we have that $[\theta_1 + \tilde{c}_2 \theta_2 + \tilde{c}_3 \theta_3]_{T_7}$ and $[-\theta_1 + \tilde{c}_2 \theta_2 + \tilde{c}_3 \theta_3]_{T_7}$ are diffeomorphic. \hfill \Box

**Lemma 4.19.** If $c_2 \cdot c_3 = 0$ and $c_2 + c_3 \neq 0$, then the algebraic restriction of the form $\sum_{k=1}^{7} c_k \theta_k |_{T_7}$ can be reduced by a symmetry of $T_7$ to an algebraic restriction $[\tilde{c}_1 \theta_1 + \tilde{c}_2 \theta_2 + \tilde{c}_3 \theta_3]_{T_7}$.

**Proof of Lemma 4.19.** We use methods similar to those used above to prove that if $c_2 \cdot c_3 = 0$ and $c_2 + c_3 \neq 0$, then $A$ is diffeomorphic to $[\tilde{c}_1 \theta_1 + \tilde{c}_2 \theta_2 + \tilde{c}_3 \theta_3]_{T_7}$. If $c_3 = 0$, then $c_2 \neq 0$ and $A = [c_1 \theta_1 + c_2 \theta_2 + c_4 \theta_4 + c_5 \theta_5 + c_6 \theta_6 + c_7 \theta_7]_{T_7}$. Let

$$\mathcal{B}_t = [c_1 \theta_1 + c_2 \theta_2 + (1-t)c_4 \theta_4 + c_5 \theta_5 + (1-t)c_6 \theta_6 + (1-t)c_7 \theta_7]_{T_7} \quad \text{for } t \in [0; 1].$$

Then $\mathcal{B}_0 = A$ and $\mathcal{B}_1 = [c_1 \theta_1 + c_2 \theta_2 + c_3 \theta_3]_{T_7}$. We prove that there exists a family $\Phi_t \in \text{Symm}(T_7)$, $t \in [0; 1]$, such that

$$\Phi_t^\ast \mathcal{B}_t = \mathcal{B}_0, \quad \Phi_0 = \text{id}. \tag{4.10}$$

Let $V_t$ be a vector field defined by $d\Phi_t/dt = V_t(\Phi_t)$. Then, by differentiating (4.10), we obtain

$$\mathcal{L}_{V_t} \mathcal{B}_t = c_4 \theta_4 + c_6 \theta_6 + c_7 \theta_7. \tag{4.11}$$
We are looking for $V_i$ in the form $V_i = b_1(t)X_1 + b_2(t)X_2 + b_4(t)X_4 + b_5(t)X_5$, where $b_k(t)$ for $k = 1, 2, 4, 5$ are smooth functions $b_k : [0; 1] \rightarrow \mathbb{R}$. Then, by Proposition 4.17, (4.11) has the form
\[
\begin{bmatrix}
7c_2 & 0 & 0 & 0 \\
0 & -3c_2 & -4c_1 & 0 \\
9c_4(1 - t) & -9c_5 & 0 & 9c_2 \\
\end{bmatrix}
\begin{bmatrix}
b_1 \\
b_2 \\
b_4 \\
b_5 \\
\end{bmatrix}
= 
\begin{bmatrix}
c_4 \\
c_6 \\
c_7 \\
\end{bmatrix}.
\] (4.12)

If $c_2 \neq 0$, we can solve (4.12), and $\Phi_t$ may be obtained as a flow of vector field $V_i$. The family $\Phi_t$ preserves $T_7$, because $V_i$ is tangent to $T_7$ and $\Phi_t B_6 = A$. Using the homotopy arguments we have that $A$ is diffeomorphic to $B_1 = [c_1 \theta_1 + c_2 \theta_2 + c_5 \theta_5]_{T_7}$. By the condition $c_2 \neq 0$, we have a diffeomorphism $\Psi \in \text{Symm}(T_7)$ of the form
\[
\Psi : (x_1, x_2, x_3) \mapsto (c_2^{-3/5} x_1, c_2^{-2/5} x_2, c_2^{-2/5} x_3)
\] (4.13)
and we obtain
\[
\Psi^*(B_1) = [c_1 c_2^{-4/5} \theta_1 + \theta_2 + c_5 c_2^{-7/5} \theta_5]_{T_7} = [\tilde{c}_1 \theta_1 + \theta_2 + c_5 \theta_5]_{T_7}.
\]

If $c_2 = 0$, then $c_3 \neq 0$ and by the diffeomorphism $\Theta \in \text{Symm}(T_7)$ of the form
\[
(\Theta^* \big|_{T_7}) : (x_1, x_2, x_3) \mapsto (x_1, x_3, x_2),
\]
we obtain
\[
\Theta^*[c_1 \theta_1 + c_3 \theta_3 + c_4 \theta_4 + c_5 \theta_5 + c_6 \theta_6 + c_7 \theta_7]_{T_7} = [-c_1 \theta_1 + c_3 \theta_2 + c_4 \theta_5 + c_5 \theta_4 - c_6 \theta_6 - c_7 \theta_7]_{T_7},
\]
and we may now use the homotopy method.

\textbf{Lemma 4.20.} If $c_2 = c_3 = 0$, $c_1 \neq 0$ and $(c_4, c_5) \neq (0, 0)$, then the algebraic restriction of the form $\sum_{k=1}^7 c_k \theta_k |_{T_7}$ can be reduced by a symmetry of $T_7$ to an algebraic restriction $[\theta_1 + c_3 \theta_4 + c_5 \theta_5]_{T_7}$.

\textbf{Lemma 4.21.} If $c_1 \neq 0$ and $c_2 = c_3 = c_4 = c_5 = 0$, then the algebraic restriction of the form $\sum_{k=1}^7 c_k \theta_k |_{T_7}$ can be reduced by a symmetry of $T_7$ to an algebraic restriction $[\theta_1 + c_5 \theta_7]_{T_7}$.

\textbf{Lemma 4.22.} If $c_1 = c_2 = c_3 = 0$ and $(c_4, c_5) \neq (0, 0)$, then the algebraic restriction of the form $\sum_{k=1}^7 c_k \theta_k |_{T_7}$ can be reduced by a symmetry of $T_7$ to an algebraic restriction $[\theta_4 + c_5 \theta_5 + c_6 \theta_6]_{T_7}$.

\textbf{Lemma 4.23.} If $c_1 = \cdots = c_5 = 0$ and $c_6 \neq 0$, then the algebraic restriction $A = \sum_{k=1}^7 c_k \theta_k |_{T_7}$ can be reduced by a symmetry of $T_7$ to an algebraic restriction $[\theta_6 + c_7 \theta_7]_{T_7}$.

\textbf{Lemma 4.24.} If $c_1 = \cdots = c_6 = 0$ and $c_7 \neq 0$, then the algebraic restriction $A = \sum_{k=1}^7 c_k \theta_k |_{T_7}$ can be reduced by a symmetry of $T_7$ to an algebraic restriction $[\theta_7]_{T_7}$.

The proofs of Lemmas 4.20–4.24 are similar and are based on Table 12.

Statement (ii) of Theorem 4.11 follows from conditions in the proof of part (i), and statement (iii) follows from Theorem 3.5, which was proved in § 4.5.

Now we prove that the parameters $c$, $c_1$, $c_2$ are moduli in the normal forms. The proofs are very similar in all cases. We consider as an example the normal form with...
two parameters \([c_1\theta_1 + \theta_2 + c_2\theta_3]_{T_7}\). From Table 12 we see that the tangent space to the orbit of \([c_1\theta_1 + \theta_2 + c_2\theta_3]_{T_7}\) at \([c_1\theta_1 + \theta_2 + c_2\theta_3]_{T_7}\) is spanned by the linearly independent algebraic restrictions \([4c_1\theta_1 + \theta_2 + 5c_2\theta_3]_{T_7}\), \([\theta_4]_{T_7}\), \([\theta_5]_{T_7}\), \([\theta_6]_{T_7}\) and \([\theta_7]_{T_7}\). Hence, the algebraic restrictions \([\theta_1]_{T_7}\) and \([\theta_3]_{T_7}\) do not belong to it. Therefore, the parameters \(c_1\) and \(c_2\) are independent moduli in the normal form \([c_1\theta_1 + \theta_2 + c_2\theta_3]_{T_7}\).

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**References**
