SINGULARITIES OF AFFINE EQUIDISTANTS: PROJECTIONS AND CONTACTS

W. DOMITRZ, P. DE M. RIOS, AND M. A. S. RUAS

Abstract. Using standard methods for studying singularities of projections and of contacts, we classify the stable singularities of affine $\lambda$-equidistants of $n$-dimensional closed submanifolds of $\mathbb{R}^q$, for $q \leq 2n$, whenever $(2n, q)$ is a pair of nice dimensions [12].

1. Introduction

When $M$ is a smooth closed curve on the affine plane $\mathbb{R}^2$, the set of all midpoints of chords connecting pairs of points on $M$ with parallel tangent vectors is called the Wigner caustic of $M$, or the area evolute of $M$, or still, the affine $1/2$-equidistant of $M$, denoted $E_{1/2}(M)$.

The $1/2$-equidistant is generalized to any $\lambda$-equidistant, denoted $E_{\lambda}(M)$, $\lambda \in \mathbb{R}$, by considering all chords connecting pairs of points of $M$ with parallel tangent vectors and the set of all points of these chords which stand in the $\lambda$-proportion to their corresponding pair of points on $M$. In this case, when $M$ is a curve on $\mathbb{R}^2$, the local classification of stable singularities of $E_{\lambda}(M)$ is well known [2, 5].

The definition of the affine $\lambda$-equidistant of $M$ is generalized to the cases when $M$ is an $n$-dimensional closed submanifold of $\mathbb{R}^q$, with $q \leq 2n$, by considering the set of all $\lambda$-points of chords connecting pairs of points on $M$ whose direct sum of tangent spaces do not coincide with $\mathbb{R}^q$, the so-called weakly parallel pairs on $M$.

In addition to curves in $\mathbb{R}^2$, the possible stable singularities of $E_{\lambda}(M)$ have been previously studied in the general setting when $M$ is a hypersurface [5, 6], or when $M$ is a surface in $\mathbb{R}^4$ [7]. The cases of curves in $\mathbb{R}^2$ and surfaces in $\mathbb{R}^4$ have also been studied in the particular setting of Lagrangian submanifolds of affine symplectic spaces [3].

In this paper, we classify the possible stable singularities of $E_{\lambda}(M)$ in a quite more general circumstance, namely, when the double dimension of $M$, $2n$, and the dimension of the ambient affine space, $q$, form a pair of nice dimensions [12], see Theorem 5.3 below.

In order to obtain such a classification, we start in Section 2 by defining an affine $\lambda$-equidistant of $M^n \subset \mathbb{R}^q$ as the set of critical values of the $\lambda$-point map (projection)

$$\pi_{\lambda} : \mathbb{R}^q \times \mathbb{R}^q \to \mathbb{R}^q, (x^+, x^-) \mapsto \lambda x^+ + (1 - \lambda)x^-$$

restricted to $M \times M$, thus locally a map

$$\tilde{\pi}_{\lambda} : \mathbb{R}^{2n} \to \mathbb{R}^q$$

see Definition 2.8, Remark 2.9 and equation (5.2), below. Then, we also present the characterization of affine equidistants by a contact map, extending previous construction for the Wigner caustic ([14, 7]).

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In Section 3 we review the standard $\mathcal{K}$-equivalence and the classification of $\mathcal{K}$-simple singularities [10, 12], Theorem 3.9 below. Then, in Section 4 we combine the study of singularities of projections and of contacts, in view of Theorem 4.6 below ([12, 11]), with emphasis on contact reduction to rank 0 map-germs, Proposition 4.14.

Our main result is obtained in Section 5. First, in Theorem 5.2 we apply the Multijet Transversality Theorem [8] to a $\mathcal{K}$-invariant stratification of the jet space. When $(2n, q)$ is a pair of nice dimensions, the relevant strata of this stratification are the $\mathcal{K}$-simple orbits in jet space. Then, we use the results of Section 4 in the context of affine equidistants: Proposition 5.4 and Corollary 5.5, as well as equations (5.8)-(5.12). The following table summarizes our main result, Theorem 5.6, which is presented more extensively as subsection 5.1. The normal forms for the $A$-stable singularities of the map $\tilde{\pi}_\lambda$ follow the notation of [10] (see Theorem 3.9 below) for the $\mathcal{K}$-simple rank-0 contact map-germ

$$\theta_\lambda : (\mathbb{R}^k, 0) \to (\mathbb{R}^{k-(2n-q)}, 0),$$

where $k$ is the degree of parallelism of the pair of points on $M$ joined by the chord (cf. Definition 2.1 and Tables I, II, III in Theorem 3.9).

<table>
<thead>
<tr>
<th>$(n, q)$</th>
<th>Stable $E_\lambda(M), M^\alpha \subset \mathbb{R}^9$</th>
<th>Restrictions</th>
</tr>
</thead>
<tbody>
<tr>
<td>(1, 2)</td>
<td>$A_\mu$</td>
<td>$\mu \leq 2$</td>
</tr>
<tr>
<td>(2, 3)</td>
<td>$A_\mu$</td>
<td>$\mu \leq 3$</td>
</tr>
<tr>
<td>(2, 4)</td>
<td>$A_\mu, C_{\mu,2}^2$</td>
<td>$\mu \leq 4$</td>
</tr>
<tr>
<td>(3, 4)</td>
<td>$A_\mu, D_4^\pm$</td>
<td>$\mu \leq 4$</td>
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<tr>
<td>(3, 5)</td>
<td>$A_\mu, D_4^\pm, D_5^\pm, S_5$</td>
<td>$\mu \leq 5$</td>
</tr>
<tr>
<td>(3, 6)</td>
<td>$A_\mu, C_{\mu,2}^\pm, C_6$</td>
<td>$\mu \leq 6, 2 \leq \rho \leq \tau, \rho + \tau \leq 6$</td>
</tr>
<tr>
<td>(4, 5)</td>
<td>$A_\mu, D_5^\pm, D_6^\pm$</td>
<td>$\mu \leq 5$</td>
</tr>
<tr>
<td>(4, 7)</td>
<td>$A_\mu, D_5^\pm, E_6, E_7, S_6, T_7, T_7$</td>
<td>$\mu \leq 7, 4 \leq \nu \leq 7, 5 \leq \beta \leq 7$</td>
</tr>
<tr>
<td>(4, 8)</td>
<td>$A_\mu, C_{\mu,2}^\pm, C_6, C_8, F_7, F_8$</td>
<td>$\mu \leq 8, 2 \leq \rho \leq \tau, \rho + \tau \leq 8$</td>
</tr>
<tr>
<td>(5, 6)</td>
<td>$A_\mu, D_6^\pm, E_6$</td>
<td>$\mu \leq 6, 4 \leq \nu \leq 6$</td>
</tr>
</tbody>
</table>

We note that the case $M^4 \subset \mathbb{R}^6$ is absent from the table of results. This is due to the fact that $(2n = 8, q = 6)$ is not a pair of nice dimensions (see Theorem 5.3 below). Similarly, $(2n, q > 6)$ is not a pair of nice dimensions, for all $n \geq 5$. Classification of stable singularities of $E_\lambda(M)$, in these cases, lies outside the scope of this paper.

As mentioned before, the cases in the table of results when

$$(n, q) \in \{(1, 2), (2, 3), (3, 4), (4, 5), (5, 6)\}$$

correspond to hypersurfaces and have been previously studied in [5, 6], and the case $(n, q) = (2, 4)$ was partially studied in [7]. On the other hand, the results for the cases when

$$(n, q) \in \{(3, 5), (3, 6), (4, 7), (4, 8)\}$$

are entirely new.

We emphasize that, in all of the above, we are excluding the cases of vanishing chords, that is, when the $\lambda$-point of the chord connecting two points on $M$ touches $M$ because the pair of points on $M$ lies in the diagonal of $M \times M$. Such “diagonal singularities” or singularities on shell for $E_\lambda(M)$ possess additional symmetries when $\lambda = 1/2$ and these have been studied for the cases of curves on the plane and surfaces in $\mathbb{R}^4$, both in the general setting [7] and in the more particular setting of Lagrangian submanifolds of affine symplectic space [4]. In this paper, we don’t study such singularities on shell for $E_\lambda(M)$. 
2. Affine equidistants

2.1. Definition of affine equidistants. Let $M$ be a smooth closed $n$-dimensional submanifold of the affine space $\mathbb{R}^q$, with $q \leq 2n$. Let $a, b$ be points of $M$ and denote by

$$
\tau_{a-b} : \mathbb{R}^q \ni x \mapsto x + (a - b) \in \mathbb{R}^q
$$

the translation by the vector $(a - b)$.

**Definition 2.1.** A pair of points $a, b \in M$ ($a \neq b$) is called a weakly parallel pair if

$$
\text{codim}(T_a M + \tau_{a-b}(T_b M)) \neq \mathbb{R}^q.
$$

If $k = n$ the pair $a, b \in M$ is called strongly parallel, or just parallel. We also refer to $k$ as the degree of parallelism of the pair $(a, b)$ and denote it by $\text{deg}(a, b)$. The degree of parallelism and the codimension of parallelism are related in the following way:

$$
2n - \text{deg}(a, b) = q - \text{codim}(a, b).
$$

**Definition 2.2.** A chord passing through a pair $a, b$, is the line

$$
l(a, b) = \{ x \in \mathbb{R}^q | x = \lambda a + (1 - \lambda)b, \lambda \in \mathbb{R} \}.
$$

**Definition 2.3.** For a given $\lambda$, an affine $\lambda$-equidistant of $M$, $E_\lambda(M)$, is the set of all $x \in \mathbb{R}^q$ such that $x = \lambda a + (1 - \lambda)b$, for all weakly parallel pairs $a, b \in M$. $E_\lambda(M)$ is also called a (affine) momentary equidistant of $M$. Whenever $M$ is understood, we write $E_\lambda$ for $E_\lambda(M)$.

Note that, for any $\lambda$, $E_\lambda(M) = E_{1-\lambda}(M)$ and in particular $E_0(M) = E_1(M) = M$. Thus, the case $\lambda = 1/2$ is special:

**Definition 2.4.** $E_{1/2}(M)$ is called the Wigner caustic of $M$ [2, 14].

2.2. Characterization of affine equidistants by projection. Consider the product affine space: $\mathbb{R}^q \times \mathbb{R}^q$ with coordinates $(x_+, x_-)$ and the tangent bundle to $\mathbb{R}^q$: $T\mathbb{R}^q = \mathbb{R}^q \times \mathbb{R}^q$ with coordinate system $(x, \dot{x})$ and standard projection $\pi : T\mathbb{R}^q \ni (x, \dot{x}) \mapsto x \in \mathbb{R}^q$.

**Definition 2.5.** For $\lambda \in \mathbb{R}$, a $\lambda$-chord transformation

$$
\Gamma_\lambda : \mathbb{R}^q \times \mathbb{R}^q \to T\mathbb{R}^q, \ (x^+, x^-) \mapsto (x, \dot{x})
$$

is a linear diffeomorphism defined by the $\lambda$-point equation:

$$
x = \lambda x^+ + (1 - \lambda)x^-,
$$

for the $\lambda$-point $x$, and a chord equation:

$$
\dot{x} = x^+ - x^-.
$$

**Remark 2.6.** For our purposes, the choice (2.4) for a chord equation is not unique, but is the simplest one. Among other possibilities, the choice $\dot{x} = \lambda x^+ - (1 - \lambda)x^-$ is particularly well suited for the study of affine equidistants of Lagrangian submanifolds in symplectic space [3].
Now, let $M$ be a smooth closed $n$-dimensional submanifold of the affine space $\mathbb{R}^q$ ($2n \geq q$) and consider the product $M \times M \subset \mathbb{R}^q \times \mathbb{R}^q$. Let $\mathcal{M}_\lambda$ denote the image of $M \times M$ by a $\lambda$-chord transformation,

$$\mathcal{M}_\lambda = \Gamma_\lambda(M \times M),$$

which is a $2n$-dimensional smooth submanifold of $T\mathbb{R}^q$.

Then we have the following general characterization:

**Theorem 2.7** ([3]). The set of critical values of the standard projection $\pi : T\mathbb{R}^q \to \mathbb{R}^q$ restricted to $\mathcal{M}_\lambda$ is $E_\lambda(M)$.

**Definition 2.8.** For $\lambda \in \mathbb{R}$, the $\lambda$-point map is the projection

$$\pi_\lambda : \mathbb{R}^q \times \mathbb{R}^q \to \mathbb{R}^q, \quad (x^+, x^-) \mapsto x = \lambda x^+ + (1 - \lambda)x^-.$$

**Remark 2.9.** Because $\pi_\lambda = \pi \circ \Gamma_\lambda$ we can rephrase **Theorem 2.7**: the set of critical values of the projection $\pi_\lambda$ restricted to $M \times M$ is $E_\lambda(M)$.

2.3. Characterization of affine equidistants by contact. In the literature, if $M \subset \mathbb{R}^2$ is a smooth curve, the Wigner caustic $E_{1/2}(M)$ has been described in various ways. A particular description says that, if $R_a : \mathbb{R}^2 \to \mathbb{R}^2$ denotes reflection through $a \in \mathbb{R}^2$, then $a \in E_{1/2}(M)$ when $M$ and $R_a(M)$ are not transversal [2, 14]. This description has also been used in [14] for the case of Lagrangian surfaces in symplectic $\mathbb{R}^4$ and, more recently [7], for the case of general surfaces in $\mathbb{R}^4$.

We now generalize this description for every $\lambda$-equidistant of submanifolds of more arbitrary dimensions.

**Definition 2.10.** For $\lambda \in \mathbb{R} \setminus \{0, 1\}$, a $\lambda$-reflection through $a \in \mathbb{R}^q$ is the map

$$R_\lambda^a : \mathbb{R}^q \to \mathbb{R}^q, \quad x \mapsto R_\lambda^a(x) = \frac{1}{\lambda}a - \frac{1 - \lambda}{\lambda}x.$$  \hspace{1cm} (2.5)

**Remark 2.11.** A $\lambda$-reflection through $a$ is not a reflection in the strict sense because

$$R_\lambda^a \circ R_\lambda^a \neq id : \mathbb{R}^q \to \mathbb{R}^q,$$

instead,

$$R_\lambda^a \circ R_\lambda^a = id : \mathbb{R}^q \to \mathbb{R}^q,$$

so that, if $a = a_\lambda = \lambda a^+ + (1 - \lambda)a^-$ is the $\lambda$-point of $(a^+, a^-) \in \mathbb{R}^{2q}$,

$$R_\lambda^a(a^-) = a^+, \quad R_\lambda^a(a^+) = a^-.$$

Of course, for $\lambda = 1/2$, $R_\lambda^{a_1/2} \equiv R_a$ is a reflection in the strict sense.

Now, let $M$ be a smooth $n$-dimensional submanifold of $\mathbb{R}^q$, with $2n \geq q$, and let

$$a = a_\lambda = \lambda a^+ + (1 - \lambda)a^-$$

be the $\lambda$-point of $(a^+, a^-) \in M \times M \subset \mathbb{R}^q \times \mathbb{R}^q$. Also, let $M^+$ be a germ of submanifold $M$ around $a^+$ and $M^-$ be a germ of submanifold $M$ around $a^-$. We have:

**Proposition 2.12.** The following statements are equivalent:

(i) The $\lambda$-point $a$ belongs to $E_\lambda(M)$.

(ii) $M^+$ and $R_\lambda^a(M^-)$ are not transversal at $a^+$.

(iii) $M^-$ and $R_\lambda^{1-\lambda}(M^+)$ are not transversal at $a^-$. 
Remark 2.13. Furthermore, from Remark 2.9 we see that the study of the singularities of affine equidistants is the study of the singularities of $\pi_{\lambda}$. But this is the same as the study of the singularities at $a = 0$ of

$$(x^+, x^-) \to x^+ + \frac{1 - \lambda}{\lambda} x^- = x^+ - R_0^1(x^-).$$

In other words, the study of the singularities of $E_\lambda(M) \ni 0$ can be proceeded via the study of the contact between $M^+$ and $R_0^1(M^-)$ or, equivalently, the contact between $M^-$ and $R_0^{1-\lambda}(M^+)$.

3. $K$-equivalence

We recall some basic definitions and results (for details, see [1]).

Henceforth, $E_s$ denotes the local ring of smooth function-germs on $\mathbb{R}^s$, and $m_s$ its maximal ideal.

Definition 3.1. Map-germs $f, \tilde{f} : (\mathbb{R}^s, y_0) \to (\mathbb{R}^t, 0)$ are $K$-equivalent if there exists a diffeomorphism-germ $\phi : (\mathbb{R}^s, y_0) \to (\mathbb{R}^s, y_0)$ and a map-germ $A : (\mathbb{R}^s, y_0) \to GL(\mathbb{R}^s)$ such that $\tilde{f} = A \cdot (f \circ \phi)$.

Theorem 3.2 ([1]). For the $K$-equivalence of two map-germs it is necessary and sufficient that two ideals generated by the components of these map-germs may be mapped one to the other by an isomorphism of $E_s$ induced by a diffeomorphism-germ of the source space $(\mathbb{R}^s, y_0)$.

Definition 3.3. A map-germ $F : (\mathbb{R}^s \times \mathbb{R}^p, (y_0, z_0)) \to \mathbb{R}^t$ is a deformation of a map-germ $f : (\mathbb{R}^s, y_0) \to \mathbb{R}^t$ if $F|_{\mathbb{R}^s \times \{z_0\}} = f$, where $p$ is the number of parameters of deformation $F$.

Definition 3.4. A diffeomorphism-germ $\Phi : (\mathbb{R}^s \times \mathbb{R}^p, (y_0, z_0)) \to (\mathbb{R}^s \times \mathbb{R}^p, (y_0, z_0))$ is called fiber-preserving if $\Phi(y, z) = (Y(y, z), Z(z))$ for a smooth map-germ $Y : (\mathbb{R}^s \times \mathbb{R}^p, (y_0, z_0)) \to (\mathbb{R}^s, y_0)$ and a diffeomorphism-germ $Z : (\mathbb{R}^p, z_0) \to (\mathbb{R}^p, z_0)$. It means that $\Phi$ preserves the fibers of the projection $pr : (\mathbb{R}^s \times \mathbb{R}^p, (y_0, z_0)) \to (\mathbb{R}^p, z_0)$.

Definition 3.5. Deformations $F, \tilde{F} : (\mathbb{R}^s \times \mathbb{R}^p, (y_0, z_0)) \to (\mathbb{R}^t, 0)$ of respective map-germs $f, \tilde{f} : (\mathbb{R}^s, y_0) \to (\mathbb{R}^t, 0)$ are fiber $K$-equivalent if there is a fiber-preserving diffeomorphism-germ $\Phi : (\mathbb{R}^s \times \mathbb{R}^p, (y_0, z_0)) \to (\mathbb{R}^s \times \mathbb{R}^p, (y_0, z_0))$, i.e. $\Phi(y, z) = (Y(y, z), Z(z))$, and a map-germ $A : (\mathbb{R}^s \times \mathbb{R}^p, (y_0, z_0)) \to GL(\mathbb{R}^t)$ such that $\tilde{F} = A \cdot (F \circ \Phi)$.

Corollary 3.6. For the fiber $K$-equivalence of two deformations it is necessary and sufficient that the two ideals of $E_{s+p}$ generated by the components of these deformations may be mapped one to the other by an isomorphism of $E_{s+p}$ induced by a fiber-preserving diffeomorphism-germ of the source space $(\mathbb{R}^s \times \mathbb{R}^p, (y_0, z_0))$.

Definition 3.7. The germ $f : (\mathbb{R}^s, 0) \to (\mathbb{R}^t, 0)$ is said to be $K$-simple if its $k$-jet, for any $k$, has a neighborhood in the jet space $J^k_{0,0}(\mathbb{R}^s, \mathbb{R}^t)$ that intersects only a finite number of $K$-equivalence classes (bounded by a constant independent of $k$).

Definition 3.8. The $p$-parameter suspension of the map-germ $f : (\mathbb{R}^s, 0) \to (\mathbb{R}^t, 0)$ is the map germ

$$F : (\mathbb{R}^s \times \mathbb{R}^p, 0) \ni (y, z) \to f(y), z) \in (\mathbb{R}^t \times \mathbb{R}^p, 0).$$

Theorem 3.9 ([10]). $K$-simple map-germs $(\mathbb{R}^s, 0) \to (\mathbb{R}^t, 0)$ with $s \geq t$ belong, up to $K$-equivalence and suspension, to one of the following three lists in Tables 1-3:
Definition 3.10. A deformation

$$F : (\mathbb{R}^s \times \mathbb{R}^p, (0,0)) \to (\mathbb{R}^t, 0)$$

of a map-germ $$f : (\mathbb{R}^s, 0) \to (\mathbb{R}^t, 0)$$ is **K-versal** if any other deformation

$$\tilde{F} : (\mathbb{R}^s \times \mathbb{R}^t, (0,0)) \to (\mathbb{R}^t, 0)$$

of $$f$$ is of the form

$$\tilde{F}(y,z) = A(y,z) \cdot F(g(y,z), h(z)),$$

where $$A : \mathbb{R}^s \times \mathbb{R}^q \to GL(\mathbb{R}^t)$$, $$g : (\mathbb{R}^s \times \mathbb{R}^q, (0,0)) \to (\mathbb{R}^s, 0)$$, $$h : (\mathbb{R}^q, 0) \to (\mathbb{R}^p, 0)$$ are map-germs such that $$A(0,0)$$ is nondegenerate matrix and $$g(y,0) = y$$.

Theorem 3.11 ([1]). K-versal deformations of K-equivalent germs with the same number of parameters are fiber K-equivalent.
4. Singularities of projection and of contact

4.1. Singularities of projection. In view of Theorem 2.7, let $M$ and $\widetilde{M}$ be smooth closed $n$-dimensional submanifolds of $\mathbb{R}^q$, $q \leq 2n$, and

$$\mathcal{M}_\lambda = \Gamma_\lambda (M \times M), \quad \widetilde{\mathcal{M}}_\lambda = \Gamma_\lambda (\widetilde{M} \times \widetilde{M}),$$

where $\Gamma_\lambda$ is the $\lambda$-chord transformation.

For local classification of singularities, we introduce the definition:

**Definition 4.1.** $E_\lambda(M)$ and $E_\lambda(\widetilde{M})$ are $\lambda$-chord equivalent if there exists a fiber-preserving diffeomorphism-germ of $T\mathbb{R}^q$ that maps the germ of $\mathcal{M}_\lambda$ to the germ of $\widetilde{\mathcal{M}}_\lambda$, i.e. if the following diagram commutes (vertical arrows indicate diffeomorphism-germs):

\[
\begin{array}{ccc}
M \times M & \xrightarrow{\Gamma_\lambda |_{M \times M}} & T\mathbb{R}^q \\
\downarrow & & \downarrow \\
\widetilde{M} \times \widetilde{M} & \xrightarrow{\Gamma_\lambda |_{\widetilde{M} \times \widetilde{M}}} & T\mathbb{R}^q \\
\end{array}
\]

The $\lambda$-chord equivalence of $E_\lambda$ is a special case of equivalence of projections studied by V. Goryunov ([9], [10]), as outlined below.

**Definition 4.2.** A projection of a (smooth) submanifold $S$ from a total space $E$ to the base $B$ of the bundle $p : E \to B$ is a triple

$$S \xleftrightarrow{\iota} E \xrightarrow{p} B$$

where $\iota$ is an embedding. A projection is called a projection “onto” if the dimension of $S$ is not less than the dimension of the base $B$.

**Definition 4.3.** Two projections $S_i \hookrightarrow E_i \to B_i$ for $i = 1, 2$ are equivalent if the following diagram commutes

\[
\begin{array}{ccc}
S_1 & \xrightarrow{\iota_1} & E_1 \\
\downarrow & & \downarrow \\
S_2 & \xrightarrow{\iota_2} & E_2 \\
\end{array} \quad \begin{array}{ccc}
 & p_1 & \\
 & \downarrow & \\
 & B_1 & \\
\end{array} \quad \begin{array}{ccc}
 & p_2 & \\
 & \downarrow & \\
 & B_2 & \\
\end{array}
\]

where vertical arrows indicate diffeomorphisms.

A projection of $S$ onto $B$ defines a family of subvarieties in the fibers of the bundle $p : E \to B$ parameterized by $B$: $S_b = S \cap p^{-1}(b)$ for any $b \in B$. A germ of the projection

$$(S, q_0) \hookrightarrow (E, e_0) \to (B, b_0)$$

can be considered in a natural way as a deformation of the subvariety $S_{b_0}$.

The germ of a bundle $E \to B$ can be identified with the germ of the trivial bundle

$$\mathbb{R}^s \times \mathbb{R}^p \to \mathbb{R}^p.$$ 

A germ of an embedded smooth submanifold $S$ can be described by the germ of the variety of zeros of some mapping-germ $F : (\mathbb{R}^s \times \mathbb{R}^p, (y_0, z_0)) \to \mathbb{R}^t$. Then $S_{z_0}$ can be identified with the germ of the variety of zeros of $F|_{\mathbb{R}^s \times \{z_0\}}$. 


If deformations $F, \tilde{F} : (\mathbb{R}^s \times \mathbb{R}^p, (y_0, z_0)) \to (\mathbb{R}^r, 0)$ of map-germs $f, \tilde{f} : (\mathbb{R}^s, y_0) \to (\mathbb{R}^r, 0)$ (respectively) are fiber $K$-equivalent then the following diagram commutes ($\Phi, Z$ indicate diffeomorphism-germs and $pr$ indicate the projection):

\[
\begin{array}{ccc}
F^{-1}(0) & \to & \mathbb{R}^s \times \mathbb{R}^p \\
\downarrow & & \downarrow \Phi \\
\tilde{F}^{-1}(0) & \to & \mathbb{R}^s \times \mathbb{R}^p \\
& & \downarrow Z \\
& & \downarrow pr
\end{array}
\]

If the ideal of function-germs vanishing on $F^{-1}(0)$ is generated by the components of $F$, then by Corollary 3.6 the inverse result is also true.

We remind that the group $A = \text{Diff}(\mathbb{R}^m, 0) \times \text{Diff}(\mathbb{R}^p, 0)$ acts on map-germs $(\mathbb{R}^m, 0) \to (\mathbb{R}^p, 0)$ by composition on source and target, with corresponding definitions for $A$-equivalent and $A$-simple (refer to Definitions 3.1 and 3.7 for the group $K$). Then, from the above we have the following results:

**Proposition 4.4** ([9, 10]). $F$ and $\tilde{F}$ are fiber $K$-equivalent if and only if the projections of $F^{-1}(0)$ and $\tilde{F}^{-1}(0)$ onto $\mathbb{R}^p$ are $A$-equivalent.

**Theorem 4.5** ([9]). If the germ of a projection $(F^{-1}(0), (0, 0)) \to (\mathbb{R}^s \times \mathbb{R}^p, (0, 0)) \to (\mathbb{R}^p, 0)$ is $A$-simple then $f = F |_{\mathbb{R}^s \times \{0\}}$ is $K$-simple.

**Theorem 4.6** ([11, 12]). The map-germ $F : \mathbb{R}^s \times \mathbb{R}^p \to \mathbb{R}^t$ is a $K$-versal deformation of a rank-0 map-germ $f : \mathbb{R}^s \to \mathbb{R}^t$ of finite $K$-codimension if and only if the projection-germ of $F^{-1}(0)$ onto $\mathbb{R}^p$ is $A$-stable (infinitesimally stable).

By Theorems 4.5 and 4.6, in order to classify stable singularities of projections one considers deformations of three classes of singularities: simple singularities of hypersurfaces (Table 1), simple singularities of curves in a 3-dimensional space (Table 3), simple singularities of a multiple point on a plane (Table 2). We are interested in projections "onto" when the projected submanifold $S = F^{-1}(0)$ is smooth and the dimension of the base $B$ of the bundle is greater than 1.

In order to see in a more clear way how these three tables are applied to the classification of singularities of affine equidistants, we now turn to the contact viewpoint.

**4.2. Singularities of contact.** Let $N_1, N_2$ be germs at $x$ of smooth $n$-dimensional submanifolds of the space $\mathbb{R}^q$, with $2n \geq q$. We describe $N_1, N_2$ in the following way:

- $N_1 = f^{-1}(0)$, where $f : (\mathbb{R}^q, x) \to (\mathbb{R}^q, 0)$ is a submersion-germ,
- $N_2 = g(\mathbb{R}^n)$, where $g : (\mathbb{R}^n, 0) \to (\mathbb{R}^q, x)$ is an embedding-germ.

Let $\tilde{N}_1, \tilde{N}_2$ be another pair of germs at $\tilde{x}$ of smooth $n$-dimensional submanifolds of the space $\mathbb{R}^q$, described in the same way.

**Definition 4.7.** The contact of $N_1$ and $N_2$ at $x$ is of the same contact-type as the contact of $\tilde{N}_1$ and $\tilde{N}_2$ at $\tilde{x}$ if there exists a diffeomorphism-germ $\Phi : (\mathbb{R}^q, x) \to (\mathbb{R}^q, \tilde{x})$ such that $\Phi(N_1) = \tilde{N}_1$ and $\Phi(N_2) = \tilde{N}_2$. We denote the contact-type of $N_1$ and $N_2$ at $x$ by $K(N_1, N_2, x)$.

**Definition 4.8.** A contact map between submanifold-germs $N_1, N_2$ is the following map-germ $f \circ g : (\mathbb{R}^n, 0) \to (\mathbb{R}^{q-n}, 0)$.

**Theorem 4.9** ([13]). $K(N_1, N_2, x) = K(\tilde{N}_1, \tilde{N}_2, \tilde{x})$ if and only if the contact maps $f \circ g$ and $f \circ \tilde{g}$ are $K$-equivalent.
Remark 4.10. If $N_1$ and $N_2$ are transversal at $x$ then it is obvious that the contact map $f \circ g : (\mathbb{R}^n, 0) \to (\mathbb{R}^{q-n}, 0)$ is a submersion-germ or a diffeomorphism-germ (when $q = 2n$).

The interesting cases are when $N_1$ and $N_2$ are not transversal at $x_0$

$$T_{x_0}N_1 + T_{x_0}N_2 \neq T_{x_0}\mathbb{R}^q.$$

**Definition 4.11.** We say that $N_1$ and $N_2$ are $k$-tangent at $x_0$ if

$$\dim(T_{x_0}N_1 \cap T_{x_0}N_2) = k.$$

If $k$ is maximal, that is

$$k = n = \dim(T_{x_0}N_1) = \dim(T_{x_0}N_2),$$

we say that $N_1$ and $N_2$ are tangent at $x_0$.

**Remark 4.12.** In order to bring this definition into the context of affine equidistants, $E_\lambda(M)$, note that $N_1 = M^+$ and $N_2 = R_0^\lambda(M^-)$ are $k$-tangent at $0$ if and only if $T_0M^+$ and $T_0M^-$ are $k$-parallel, where $\lambda a + (1 - \lambda)b = 0 \in E_\lambda(M)$.

If $N_1$ and $N_2$ are $k$-tangent then we can describe germs of $N_1$ and $N_2$ at $0$ in the following way:

(4.1) \[ N_1 = \{(y, z, u, v) \in \mathbb{R}^q : u = \phi(y, z), \ v = \psi(y, z)\}, \]

(4.2) \[ N_2 = \{(y, z, u, v) \in \mathbb{R}^q : z = \eta(y, v), \ u = \zeta(y, v)\}, \]

where $y = (y_1, \cdots, y_k)$, $z = (z_1, \cdots, z_{n-k})$, $u = (u_1, \cdots, u_{q+k-2n})$, $v = (v_1, \cdots, v_{n-k})$ and $(y, z, u, v)$ is a coordinate system on the affine space $\mathbb{R}^q$,

$$\phi = (\phi_1, \cdots, \phi_{q+k-2n}), \quad \psi = (\psi_1, \cdots, \psi_{n-k}),$$

$$\eta = (\eta_1, \cdots, \eta_{n-k}), \quad \zeta = (\zeta_1, \cdots, \zeta_{q+k-2n}), \quad \text{and} \quad \phi_i, \psi_j, \eta_j, \zeta_i \in \mathcal{M}_\lambda^2,$$

for $i = 1, \cdots, q+k-2n$ and $j = 1, \cdots, n-k$.

Then, the contact map $\kappa_{N_1, N_2} : (\mathbb{R}^n, 0) \to (\mathbb{R}^{q-n}, 0)$ is given by:

(4.3) \[ \kappa_{N_1, N_2}(y, z) = (z - \eta(y, \psi(y, z)), \phi(y, z) - \zeta(y, \psi(y, z))). \]

From the form of $\kappa_{N_1, N_2}$ we easily obtain the following fact

**Proposition 4.13.** If $N_1$ and $N_2$ are $k$-tangent at $0$ then the corank of the contact map $\kappa_{N_1, N_2}$ is $k$.

We can interpret the contact between two $k$-tangent $n$-dimensional submanifolds $N_1, N_2$ of $\mathbb{R}^q$ as the contact between tangent $k$-dimensional submanifolds $P_{N_1}$ and $P_{N_2}$ of $N_1$ and $N_2$, respectively, in a smooth $q - 2n + 2k$-dimensional submanifold $S$ of $\mathbb{R}^q$. These submanifolds are constructed in the following way:

Let $H$ be a smooth $q+k-n$-dimensional submanifold-germ on $\mathbb{R}^q$ which contains $N_1$ and is transversal to $N_2$ at $0$. Then $P_{N_2} = H \cap N_2$ is a smooth $k$-dimensional submanifold on $N_2$.

Let $G$ be a smooth $q+k-n$-dimensional submanifold-germ on $\mathbb{R}^q$ which contains $N_2$ and is transversal to $N_1$ at $0$. Then $P_{N_1} = G \cap N_1$ is a smooth $k$-dimensional submanifold on $N_1$.

$P_{N_1}$ and $P_{N_2}$ are tangent at $0$ and they are contained in the smooth $q - 2n + 2k$-dimensional submanifold-germ $S = H \cap G$.

The contact between $N_1$ and $N_2$ at $0$ can now be described as the contact between $P_{N_1}$ and $P_{N_2}$ at $0$, which defines a rank-0 map

(4.4) \[ \kappa_{P_{N_1}, P_{N_2}} : (\mathbb{R}^k, 0) \to (\mathbb{R}^{k-(2n-q)}, 0). \]
Although in general $P_{N_1}$ and $P_{N_2}$ depend on the choices of $H$ and $G$, the contact type of $P_{N_1}$ and $P_{N_2}$ does not depend on these choices. This means that if $\tilde{N}_1, \tilde{N}_2$ is another pair of germs at 0 of smooth $n$-dimensional submanifold of $\mathbb{R}^q$, then we have the following result.

**Proposition 4.14.** $\mathcal{K}(N_1, N_2, 0) = \mathcal{K}(\tilde{N}_1, \tilde{N}_2, 0)$ if and only if

$$\mathcal{K}(P_{N_1}, P_{N_2}, 0) = \mathcal{K}(P_{\tilde{N}_1}, P_{\tilde{N}_2}, 0).$$

**Proof.** It is easy to see that in general $H$ can be described in the following way:

$$v = \psi(y, z) + A(y, z, u, v)(u - \phi(y, z)),$$

and $G$ can be described in the following way:

$$z = \eta(y, v) + B(y, z, u, v)(u - \zeta(y, v)),$$

where $A = (a_{ij})_{i=1,\ldots,q-k+2n, j=1,\ldots,n-k}$, $B = (b_{ij})_{i=1,\ldots,q-k+2n, j=1,\ldots,n-k}$, and $a_{ij}, b_{ij}$ are smooth function-germs on $\mathbb{R}^q$.

Thus $S = H \cap G$ is given by (4.5) and (4.6).

$P_{N_1}$ is given by (4.5), (4.6), and $u = \phi(y, z)$ and $P_{N_2}$ is given by (4.5), (4.6) and $u = \zeta(y, v)$.

On the other hand we can also describe $N_1$ by (4.5) and $u = \phi(y, z)$ and $N_2$ by (4.6) and $u = \zeta(y, v)$. Then it is easy to see that contact maps are the same after a suitable suspension. □

In view of Proposition 4.14, it is enough to classify the rank-0 map-germs of the form (4.4) with respect to the group $K$.

5. Stable singularities of affine equidistants

Since our goal is to classify singularities of affine equidistants of $n$-dimensional submanifold $M$ of $\mathbb{R}^q$, we substitute submanifold-germs $N_1$ and $N_2$ of the previous section by $N_1 = M^+$ and $N_2 = R_0^1(M^-)$, or equivalently by $N_1 = M^+$ and $N_2 = R_1^{1-\lambda}(M^+)$, where $M^+$ and $M^-$ are germs of $M \subset \mathbb{R}^q$ at points $a^+ \neq a^- \in M \subset \mathbb{R}^q$, such that $\lambda a^+ + (1-\lambda)a^- = 0$.

First, we state the following definition and theorem:

**Definition 5.1.** A mapping $\psi : N^m \to \mathbb{R}^q$ is locally stable at $p \in N^m$ if there exists a neighbourhood $W_p$ of $\psi$ in the space $C^\infty(N^m, \mathbb{R}^q)$ of $C^\infty$-mappings from $N^m$ into $\mathbb{R}^q$ with the Whitney $C^\infty$-topology, and neighbourhoods $U_p$ around $p$ and $V_p$ around $\psi(p)$ such that for all $\phi \in W_p$, it follows that $\phi : U_p \to V_p$ is $A$-equivalent to $\psi : U_p \to V_p$, where $A = \text{Diff}(U_p) \times \text{Diff}(V_p)$ (see [8]).

**Theorem 5.2.** For a residual set of embeddings $\iota : M^n \to \mathbb{R}^q$ the map

$$\pi_\lambda \circ (\iota \times \iota) : M \times M \setminus \Delta \to \mathbb{R}^q$$

is locally stable whenever the pair $(2n, q)$ is a pair of nice dimensions, where $\Delta$ is the diagonal in $M \times M$.

**Proof.** From the diagram of maps

$$M \times M \xrightarrow{\iota \times \iota} \mathbb{R}^q \times \mathbb{R}^q \xrightarrow{\pi_\lambda} \mathbb{R}^q,$$

we obtain the diagram of $r$-jet maps

$$j_r(\iota \times \iota) : M \times M \to J^r(M \times M, \mathbb{R}^q \times \mathbb{R}^q) \xrightarrow{(\pi_\lambda)_*} J^r(M \times M, \mathbb{R}^q).$$

A typical fiber of $J^r(M \times M, \mathbb{R}^q)$ is $J_q^r(M \times M, \mathbb{R}^q)$, the space of (degree $\leq r$)-polynomial map-germs $\mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^q$, vanishing at 0.
Let \{W_1, \ldots, W_s\} be the finite set of all \(K\) simple orbits in \(J^r(M \times M, \mathbb{R}^q)\); let \{W_{s+1}, \ldots, W_t\} be a finite stratification of the complement of the union of simple orbits \(W_1 \cup \ldots \cup W_s\). This stratification exists because these are semialgebraic sets. We denote by \(S = \{W_j\}_{1 \leq j \leq t}\) the resulting stratification of \(J^r(M \times M, \mathbb{R}^q)\). Because \((\pi\lambda)\) is a submersion, \((\pi\lambda)^{-1}W_j = W_j^*\) is a submanifold of \(J^*(M \times M, \mathbb{R}^q \times \mathbb{R}^q)\), for all \(j = 1, \ldots, t\), so that \(S^* = \{W_j^*\}_{1 \leq j \leq t}\) is a stratification of this space.

Furthermore,

\[(5.1)\quad j^*(\iota \times \iota) \cap S^* \iff j^*(\pi\lambda \circ (\iota \times \iota)) \cap S,\]

where transversality to \(S\) (respectively to \(S^*\)) means transversality of \(j^*(\pi\lambda \circ (\iota \times \iota))\) to each stratum of the corresponding stratification.

On the other hand, under the natural identification

\[j^*(\iota \times \iota)|_{M \times M, \lambda} \simeq 2j^*\iota \subset 2J^r(M, \mathbb{R}^q),\]

where \(2J^r(M, \mathbb{R}^q)\) is the space of double \(r\)-jets, we can apply the Multijet Transversality Theorem [8] to get that, for each \(W_j^*\) in \(2J^r(M, \mathbb{R}^q)\), the set of immersions

\[\mathcal{R}_{W_j} = \{\iota : M \to \mathbb{R}^q | 2j^*\iota \cap W_j^*\}\]

is residual. Then, the set

\[\mathcal{R} = \cap_{j=1}^t \mathcal{R}_{W_j}\]

is also residual.

Now, it follows from equation (5.1) that \(j^*(\pi\lambda \circ (\iota \times \iota)) \cap W_j\), for all \(\iota \in \mathcal{R}\), for all \(j = 1, \ldots, t\). When \((2n, q)\) is a pair of nice dimensions, this implies that \(j^*(\pi\lambda \circ (\iota \times \iota))\) is transversal to all \(K\) orbits in \(J^r(M \times M, \mathbb{R}^q)\), which says that this mapping is locally stable (see [8, 12]).

**Theorem 5.3 ([12]).** The nice dimensions for pairs \((2n, q)\) are:

(i) \(n < q = 2n, \ n \leq 4\)

(ii) \(n < q = 2n - 1, \ n \leq 4\)

(iii) \(n < q = 2n - 2, \ n \leq 3\)

(iv) \(n < q \leq 2n - 3, \ q \leq 6\)

Thinking locally, denote two distinct germs of embedding \(\iota : M^n \to \mathbb{R}^q\) by

\[\iota^+ : (\mathbb{R}^n, 0) \to (\mathbb{R}^q, a^+) \quad \text{and} \quad \iota^- : (\mathbb{R}^n, 0) \to (\mathbb{R}^q, a^-),\]

and by

\[(5.2)\quad \tilde{\pi}\lambda = \pi\lambda \circ (\iota^+ \times \iota^-) : (\mathbb{R}^{2n}, 0) \to (\mathbb{R}^q, 0),\]

the restriction of \(\pi\lambda\) to \(M^+ \times M^-\). Then, recalling the notation of (4.1)-(4.2), \(\tilde{\pi}\lambda\) is given by

\[(5.3)\quad \tilde{\pi}\lambda : (y, z, \tilde{y}, v) \to (\tilde{\pi}\lambda^1(y, \tilde{y}), \tilde{\pi}\lambda^2(z, \tilde{y}, v), \tilde{\pi}\lambda^3(y, z, \tilde{y}, v), \tilde{\pi}\lambda^4(y, z, v))\]

where \(y, \tilde{y} \in \mathbb{R}^k, \ z, v \in \mathbb{R}^{n-k}\), and

\[(5.4)\quad \tilde{\pi}\lambda^1(y, \tilde{y}) = \lambda y + (1 - \lambda)\tilde{y},\]

\[(5.5)\quad \tilde{\pi}\lambda^2(z, \tilde{y}, v) = \lambda z + (1 - \lambda)\eta(\tilde{y}, v),\]

\[(5.6)\quad \tilde{\pi}\lambda^3(y, z, \tilde{y}, v) = \lambda \phi(y, z) + (1 - \lambda)\zeta(\tilde{y}, v),\]

\[(5.7)\quad \tilde{\pi}\lambda^4(y, z, v) = \lambda \psi(y, z) + (1 - \lambda)v.\]

Let

\[\kappa\lambda : (\mathbb{R}^n, 0) \to (\mathbb{R}^{q-n}, 0)\]

denote the the contact-map between \(M^+\) and \(\mathcal{R}\lambda^0(M^-)\). We have:
Proposition 5.4. Local rings $\frac{E_{2n}}{\pi_{A}^{n}(m_{q})}$ and $\frac{E_{n}}{\kappa_{A}^{n}(m_{q-n})}$ are isomorphic.

Proof. From (5.3), we have that

$$\frac{E_{2n}}{\pi_{A}^{n}(m_{q})} \simeq \frac{E_{(y,z,\tilde{y},\tilde{v})}}{(\tilde{\pi}_{n}^{4}(y,\tilde{y}), \tilde{\pi}_{n}^{4}(z,\tilde{y},v), \tilde{\pi}_{n}^{4}(y,z,\tilde{y},v), \tilde{\pi}_{n}^{4}(y,z,v))}$$

so that, using (5.4)-(5.7), this is isomorphic to

$$\frac{E_{(y,z)}}{(\pi_{n}^{4}(y,z), \psi(y,z))}$$

and, using (4.3) for $N = M^{+}$ and $N_{2} = R_{0}^{n}(M^{-})$, we see that the above local ring is isomorphic to $\frac{E_{n}}{\kappa_{A}^{n}(m_{q-n})}$. $\square$

On the other hand, we remind from Remark 4.12 that $k$ is the degree of tangency of $M^{+}$ and $R_{0}^{n}(M^{-})$, and therefore $k$ is the degree of parallelism of $T_{0}^{+}M^{+}$ and $T_{0}^{-}M^{-}$, where

$$\lambda a^{+} + (1 - \lambda)a^{-} = 0 \in E_{A}(M),$$

so that, denoting by

$$\theta_{A} : (\mathbb{R}^{k},0) \rightarrow (\mathbb{R}^{k-2n},0)$$

the reduced (rank-0) contact map $\theta_{A} = \kappa_{P_{N_{1}},P_{N_{2}}}$, for $N_{1} = M^{+}$ and $N_{2} = R_{0}^{n}(M^{-})$, from Proposition 4.14 we have the following

Corollary 5.5. The local rings $\frac{E_{n}}{\kappa_{A}^{n}(m_{q-n})}$ and $\frac{E_{k}}{\kappa_{A}^{n}(m_{q-2n})}$ are isomorphic.

Thus, by Theorems 4.6 and 5.2, Proposition 5.4 and Corollary 5.5, for the local classification of stable singularities of affine equidistants, we need to determine every rank-0 $K$-simple map-germ

$$(5.8) \quad \theta_{A} : (\mathbb{R}^{k},0) \rightarrow (\mathbb{R}^{l},0),$$

that admits a $K$-versal deformation $F_{A} : \mathbb{R}^{k} \times \mathbb{R}^{q} \rightarrow \mathbb{R}^{l}$, so that

$$(5.9) \quad \pi_{A} : (F_{A})^{-1}(0) = (\mathbb{R}^{2n},0) \rightarrow (\mathbb{R}^{q},0)$$

is an $A$-stable map. Here, $\theta_{A} = \kappa_{P_{N_{1}},P_{N_{2}}}$, for $N_{1} = M^{+}$ and $N_{2} = R_{0}^{n}(M^{-})$, and $\pi_{A}$ is the restriction of $\pi_{A}$ to $M^{+} \times M^{-}$, so that

$$(5.10) \quad l = k - (2n - q), \quad 1 \leq k \leq n, \quad 2n \geq q > n,$$

for any pair $(2n, q)$ in the nice dimensions (Theorem 5.3).

In other words, we unfold the map-germ $\theta_{A}$ with $m$ parameters,

$$(5.11) \quad \tilde{\pi}_{A} : (\mathbb{R}^{m} \times \mathbb{R}^{k},0) \rightarrow (\mathbb{R}^{m} \times \mathbb{R}^{l},0), \quad (w, y) \rightarrow (w, u(w,y)),$$

where $m = 2n - k$, so that $\tilde{\pi}_{A}$ is $A$-stable. Thus, in each case, we look for the rank-0 $K$-simple map-germs $\theta_{A}$ that can be unfolded with $m = 2n - k$ parameters so that its $K_{A}$-codimension $\mu$ is such that

$$(5.12) \quad \mu \leq l + m = q.$$

The list of $K$-simple map-germs $\theta_{A}$ is presented in Tables 1, 2 and 3, in section 2 above. Thus, for classifying the stable singularities of affine equidistants of smooth submanifolds $M^{n} \subset \mathbb{R}^{q}$, all we have to do is read those Tables with respect to the numbers $k, l$ and $\mu$, subject to conditions (5.10) and (5.12) for each pair $(2n, q)$ in the nice dimensions.

In this way, we arrive at our main result, as follows.
5.1. All possible stable singularities in the nice dimensions. First, remind the definition of \(k\)-parallelism, cf. (2.1). Then, we have:

**Theorem 5.6.** Let \(M^n \subset \mathbb{R}^q\) be a smooth closed submanifold of the affine space, such that \(2n \geq q\) and \((2n, q)\) is a pair of nice dimensions, as listed in Theorem 5.3. Then, the possible stable singularities of the \(\lambda\)-affine equidistant \(E_\lambda(M) \subset \mathbb{R}^q\) are listed case by case, as below.

**Curves:**

In this case, we have curves in \(\mathbb{R}^2\) and the rank-0 contact map is \(\theta_\lambda : \mathbb{R} \rightarrow \mathbb{R}, \mu \leq 2\). From Table 1, the stable singularities of affine equidistants can be of type \(A_1\) and \(A_2\).

**Surfaces:**

(1) \(M^2 \subset \mathbb{R}^3\).
- 2-parallelism. \(\theta_\lambda : \mathbb{R}^2 \rightarrow \mathbb{R}, \mu \leq 3\).
- \(E_\lambda(M)\) with stable singularities of types \(A_1, A_2\) and \(A_3\).

(2) \(M^2 \subset \mathbb{R}^4\).
- (i) 1-parallelism. \(\theta_\lambda : \mathbb{R} \rightarrow \mathbb{R}, \mu \leq 4\).
  - \(E_\lambda(M)\) with stable singularities of types \(A_1, A_2, A_3\) and \(A_4\).
- (ii) 2-parallelism. \(\theta_\lambda : \mathbb{R}^2 \rightarrow \mathbb{R}^2, \mu \leq 4\).
  - \(E_\lambda(M)\) with stable singularities of types \(C_{2,2}^\pm\).

3-manifolds:

(1) \(M^3 \subset \mathbb{R}^4\).
- 3-parallelism. \(\theta_\lambda : \mathbb{R}^3 \rightarrow \mathbb{R}, \mu \leq 4\).
  - \(E_\lambda(M)\) with stable singularities of types \(A_1, ..., A_5, D_4^\pm, D_5^\pm, D_6^\pm\).

(2) \(M^3 \subset \mathbb{R}^5\).
- (i) 2-parallelism. \(\theta_\lambda : \mathbb{R}^2 \rightarrow \mathbb{R}, \mu \leq 5\).
  - \(E_\lambda(M)\) with stable singularities of types \(A_1, ..., A_5, D_4^\pm, D_5^\pm\).
- (ii) 3-parallelism. \(\theta_\lambda : \mathbb{R}^3 \rightarrow \mathbb{R}^2, \mu \leq 5\).
  - \(E_\lambda(M)\) with stable singularities of types \(S_5\).

(3) \(M^3 \subset \mathbb{R}^6\).
- (i) 1-parallelism. \(\theta_\lambda : \mathbb{R} \rightarrow \mathbb{R}, \mu \leq 6\).
  - \(E_\lambda(M)\) with stable singularities of types \(A_1, ..., A_6\).
- (ii) 2-parallelism. \(\theta_\lambda : \mathbb{R}^2 \rightarrow \mathbb{R}^2, \mu \leq 6\).
  - \(E_\lambda(M)\) with stable singularities of types \(C_{2,2}^\pm, C_{2,3}^\pm, C_{2,4}^\pm, C_{3,3}^\pm, C_6\).
- (iii) 3-parallelism. No stable singularities for \(E_\lambda(M)\).

4-manifolds:

(1) \(M^4 \subset \mathbb{R}^5\).
- 4-parallelism. \(\theta_\lambda : \mathbb{R}^4 \rightarrow \mathbb{R}, \mu \leq 5\).
  - \(E_\lambda(M)\) with stable singularities of types \(A_1, ..., A_5, D_4^\pm, D_5^\pm\).
(2) $M^4 \subset \mathbb{R}^6$: The map $\tilde{\pi}_\lambda : \mathbb{R}^8 \to \mathbb{R}^6$ is not in nice dimensions.

(3) $M^4 \subset \mathbb{R}^7$.
(i) 2-parallelism. $\theta_\lambda : \mathbb{R}^2 \to \mathbb{R}$, $\mu \leq 7$.
$E_\lambda(M)$ with stable singularities of types $A_1, ..., A_7, D^+_4, ..., D^+_7, E_6, E_7$.
(ii) 3-parallelism. $\theta_\lambda : \mathbb{R}^3 \to \mathbb{R}^2$, $\mu \leq 7$.
$E_\lambda(M)$ with stable singularities of types $S_5, S_6, S_7, T_7, \tilde{T}_7$.
(iii) 4-parallelism. No stable singularities for $E_\lambda(M)$.

(4) $M^4 \subset \mathbb{R}^8$.
(i) 1-parallelism. $\theta_\lambda : \mathbb{R} \to \mathbb{R}$, $\mu \leq 8$.
$E_\lambda(M)$ with stable singularities of types $A_1, ..., A_8$.
(ii) 2-parallelism. $\theta_\lambda : \mathbb{R}^2 \to \mathbb{R}^2$, $\mu \leq 8$.
$E_\lambda(M)$ with stable singularities of types $C^+_{2,2}, C^+_{2,3}, C^+_{3,3}, C^+_{3,4}, C^+_{3,5}, C^+_{4,4}, C_6, C_8, F_7, F_8$.
(iii) 3-parallelism, 4-parallelism. No stable singularities for $E_\lambda(M)$.

5-manifolds:

(1) $M^5 \subset \mathbb{R}^6$.
5-parallelism. $\theta_\lambda : \mathbb{R}^5 \to \mathbb{R}$, $\mu \leq 6$.
$E_\lambda(M)$ with stable singularities of types $A_1, ..., A_6, D^+_4, D^+_5, D^+_6, E_6$.

(2) For all other embeddings $M^5 \subset \mathbb{R}^q$, no map $\tilde{\pi}_\lambda$ in nice dimensions.

$n$-manifolds, $n \geq 6$: No map $\tilde{\pi}_\lambda$ in nice dimensions.

References
